Lecture 7: Amortized Analysis

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601.433/633 Introduction to Algorithms

Introduction

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Data structures: *sequence* of operations!

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Last time: analyzed the (worst-case) cost of each operation.
What about (worst-case) cost of *sequence* of operations?

Definition & Example

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Example: 100 operations of cost $\mathbf{1}$, then $\mathbf{1}$ operation of cost $\mathbf{100}$

- Normal worst-case analysis: $\mathbf{100}$
- Amortized cost: $\mathbf{200/101 \approx 2}$

Definition & Example

Definition

The *amortized cost* of a sequence of n operations is the total cost of the sequence divided by n.

"Average cost per operation" (but no randomness!)

Example: 100 operations of cost 1, then 1 operation of cost 100

- Normal worst-case analysis: 100
- Amortized cost: $200/101 \approx 2$

If we care about total time (e.g., using data structure in larger algorithm) then worst-case too pessimistic

Amortized Algorithm

Still want worst-case, but worst-case over *sequences* rather than single operations.

Maybe only possible way to have an expensive operation is to have a bunch of cheap operations: amortized cost always small!

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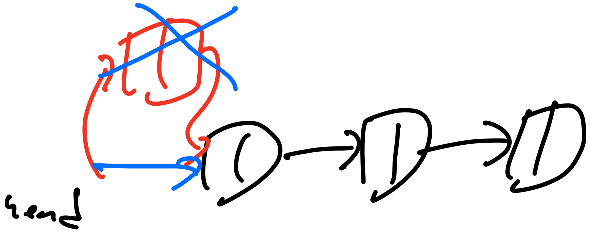
Definition

If the amortized cost of *every* sequence of $\mathbf{n}$ operations is at most $\mathbf{f(n)}$, then the *amortized cost* or *amortized complexity* of the algorithm is at most $\mathbf{f(n)}$.

Example: Stack From Array

Stack Using Array



Stack:

- Last In First Out (LIFO)
- Push: add element to stack
- Pop: Remove the most recently added element.

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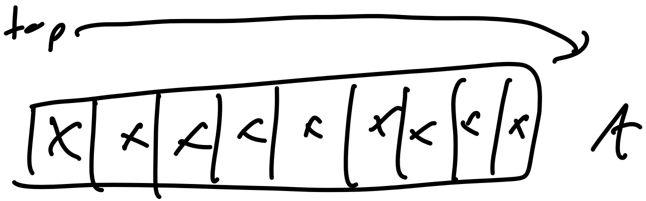
Building a stack with an array `A`:

- Initialize: `top = 0`
- Push($\mathbf{x}$): `A[top] =` $\mathbf{x}$`; top++`
- Pop: `top--; Return A[top]`



End of Array

What if array is full (**n** elements)?

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What if array is full ($\mathbf{n}$ elements)?

Make new, bigger array, copy old array over

- Cost: free to create new array, each copy costs $\mathbf{1}$
- Worst case: a single Push could cost $\mathbf{\Omega(n)}$!

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New array has size $\mathbf{n+1}$:

- Sequence of $\mathbf{n}$ Push operations. Total cost: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$.
- Amortized cost: $\mathbf{\Theta(n)}$ (same as worst single operation!)

Better Idea

Instead of increasing from $\mathbf{n}$ to $\mathbf{n+1}$:

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Consider *any* sequence of $\mathbf{n}$ operations.

- Have to double when array has size $\mathbf{2, 4, 8, 16, 32, 64, \ldots, \lfloor \log n \rfloor}$
- *Total* time spent doubling: at most $\sum_{i=1}^{\lfloor \log n \rfloor} 2^i \leq 2n = \Theta(n)$
- Any operation that doesn't cause a doubling costs $\mathbf{O(1)}$
- Total cost at most $\mathbf{O(n) + n \cdot O(1) = O(n)}$
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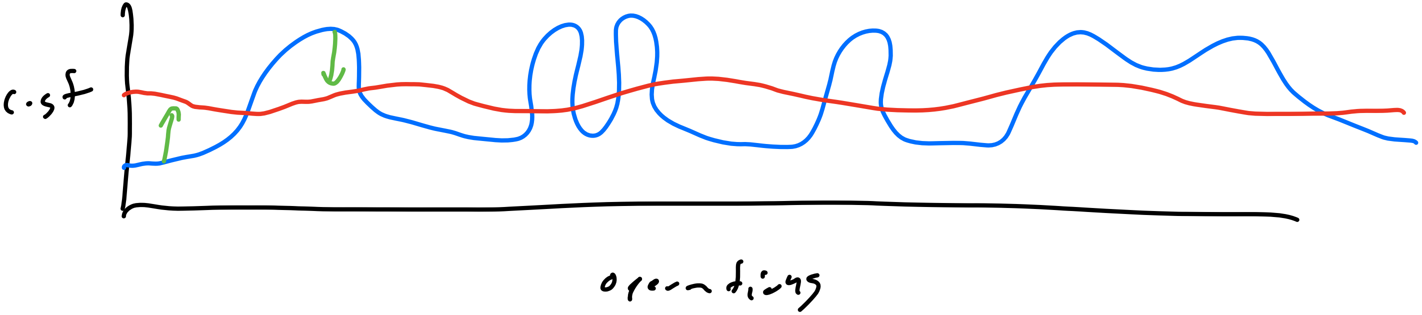
Amortized analysis explains why it's better to double than add $\mathbf{1}$!

More Complicated Analysis: Piggy Banks and Potentials

Basic Bank: Informal

Can be hard to give good bound directly on total cost.

- Lots of variance: some operations very expensive, some very cheap.
- Idea: "smooth out" the operations.
- "Pay more" for cheap operations, "pay less" for expensive ops.



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- Cheap operation: add to the bank
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Charge cheap operations more, use extra to pay for expensive operations

Basic Bank: Formal

Bank $\mathbf{L}$.

- Initially $\mathbf{L} = \mathbf{0}$
- $\mathbf{L_i}$ = value of bank ofter operation $\mathbf{i}$ (so $\mathbf{L_0} = \mathbf{0}$).

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Operation $\mathbf{i}$:

- Cost $\mathbf{c_i}$
- "Amortized cost" $\mathbf{c'_i} = \mathbf{c_i} + \mathbf{\Delta L} = \mathbf{c_i} + \mathbf{L_i} - \mathbf{L_{i-1}}$

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Total cost of sequence:

telescoping sum

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \left(c'_i + L_{i-1} - L_i\right) = \sum_{i=1}^{n} c'_i + \sum_{i=1}^{n} (L_{i-1} - L_i) = \left(\sum_{i=1}^{n} c'_i\right) + L_0 - L_n$$

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So if $\mathbf{L_0} = \mathbf{0}$ and $\mathbf{L_n} \geq \mathbf{0}$ (bank not negative): $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} c'_i$.

Basic Bank: Formal

Bank L.

- Initially $L = 0$
- L_i = value of bank ofter operation i (so $L_0 = 0$).

Operation i:

- Cost c_i
- "Amortized cost" $c'_i = c_i + \Delta L = c_i + L_i - L_{i-1} \implies c_i = c'_i + L_{i-1} - L_i$

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So if $L_0 = 0$ and $L_n \geq 0$ (bank not negative): $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} c'_i$.

- If $c'_i \leq f(n)$ for all i, then "true" amortized cost $(\sum_{i=1}^{n} c_i)/n$ also at most $f(n)$!

Variants

Multiple banks

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Potential Functions:

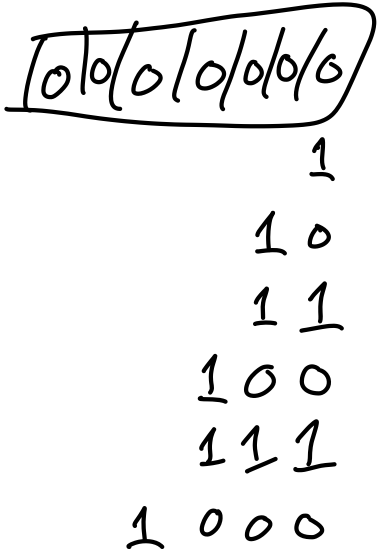
- "Bank analogy": we choose how much to deposit/withdraw.
- New analogy: "potential energy". Function of state of system.
- Rename $\mathbf{L}$ to $\mathbf{\Phi}$: all previous analysis works same!
- Sometimes easier to think about: just define once at the beginning, instead of for each operation.

Example: Binary Counter

Binary Counter

Super simple setup: binary counter stored in array **A**.

- Least significant bit in $\mathbf{A}[\mathbf{0}]$, then $\mathbf{A}[\mathbf{1}]$, ...
- Don't worry about length of array (infinite, or long enough)
- Only operation is increment.
- Costs **1** to flip any bit.



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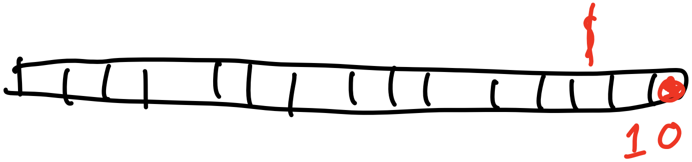
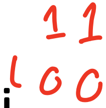
What about amortized cost?

Banks

Bank for every bit **A[i]**





Flip bit **i** from **0** to **1**: add \$ to bank for **i**
Flip bit **i** from **1** to **0**: remove \$ from bank for **i**

- No bank ever negative (induction)

Analysis

Do an increment, flips $\mathbf{k}$ bits $\implies$ true cost is $\mathbf{k}$.

- \# $\mathbf{0}$'s flipped to $\mathbf{1}$:
- \# $\mathbf{1}$'s flipped to $\mathbf{0}$:

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Global: Change in *total* bank is $-(\mathbf{k}-\mathbf{1})+\mathbf{1}=-\mathbf{k}+\mathbf{2}$
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Potential function: let $\mathbf{\Phi} = \#\mathbf{1}$'s in counter.
$\implies$ amortized cost $= \mathbf{c} + \mathbf{\Delta\Phi} = \mathbf{k} + (-\mathbf{k}+2) = 2$

Example: Simple Dictionary

Setup

Same dictionary problem as last lecture (insert, lookup).

- Can we do something simple with just arrays (no trees)?
- Give up on worst-case: try for amortized.
  - Sorted array: inserts $\mathbf{\Omega(n)}$ amortized ($\mathbf{i}$'th insert could take time $\mathbf{\Omega(i)}$)
  - Unsorted array: lookups $\mathbf{\Omega(n)}$ amortized

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Solution: array of arrays!

- $A[i]$ either empty or a *sorted* array of *exactly* 2^i elements
- No relationship between arrays

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Example: insert $1 - 11$

$$A[0] = [5]$$
$$A[1] = [2, 8]$$
$$A[2] = \emptyset$$
$$A[3] = [1, 3, 4, 6, 7, 9, 10, 11]$$

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- Binary search in each (nonempty) array
- Time at most $\sum_{i=0}^{\lceil \log n \rceil} \log(2^i) = \Theta(\log^2 n)$

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Insert($\mathbf{x}$):

- Create array $\mathbf{B = [x]}$
- $\mathbf{i = 0}$
- Otherwise: $\mathbf{i = 0}$
  - If $\mathbf{A[i]} = \varnothing$, set $\mathbf{A[i] = B}$, return.
  - Merge $\mathbf{B}$ and $\mathbf{A[i]}$ to get $\mathbf{B}$
  - Set $\mathbf{A[i]} = \varnothing$
  - $\mathbf{i}$++

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$\mathbf{A}[0] = \varnothing$

$\mathbf{A}[1] = \varnothing$

$\mathbf{A}[2] = [2, 5, 8, 12]$

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Amortized:

- Merge arrays of length $\mathbf{2^i}$ one out of every $\mathbf{2^i}$ inserts
- So after $\mathbf{n}$ inserts, have merged arrays of length $\mathbf{1}$ at most $\mathbf{n}$ times, arrays of length $\mathbf{2}$ at most $\mathbf{n/2}$ times, arrays of length $\mathbf{4}$ at most $\mathbf{n/4}$ times, ...

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- Total cost at most

$$\sum_{i=1}^{\lfloor \log n \rfloor} \frac{n}{2^{i-1}} 2^{i+1} = \Theta(n \log n)$$

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$$\sum_{i=1}^{\lfloor \log n \rfloor} \frac{n}{2^{i-1}} 2^{i+1} = \Theta(n \log n)$$

- Amortized cost at most $\Theta(\log n)$!

Multiple Operations

How do we define amortized analysis of data structures with multiple operations?

Definition

If structure supports $\mathbf{k}$ operations, say that operation type $\mathbf{i}$ has amortized cost at most $\alpha_{\mathbf{i}}$ if for every sequence which performs with at most $\mathbf{m_i}$ operations of type $\mathbf{i}$, the total cost is at most $\sum_{\mathbf{i}=1}^{\mathbf{k}} \alpha_{\mathbf{i}} \mathbf{m_i}$.

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- When analyzing multiple operations, need to use the same bank/potential for all of them!
- With multiple operations, bounds not necessarily unique. Different amortization schemes could yield different bounds, all of which are correct and non-contradictory.