Lecture 5: Sorting Lower Bound and “Linear-Time” Sorting

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September 14, 2021
601.433/633 Introduction to Algorithms

Reminders

HW2 due on Thursday!

Remember:

- Include your group members on the first page
- Typeset your solutions
- Label your pages in gradescope

Ethics policy!

Introduction

Lots of ways of sorting in $\mathbf{O(n \log n)}$ time: mergesort, heapsort, randomized quicksort, deterministic quicksort with BPFRT pivot selection, ...

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- All algorithms we've seen so far have been in this model

No: every algorithm in the comparison model must have worst-case running time $\mathbf{\Omega(n \log n)}$.

Yes: If we assume extra structure for the elements, can do sorting in $\mathbf{O(n)}$ time*

Sorting Lower Bound

Statement

Theorem

Any sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Lower bound on the number of comparisons – running time could be even worse!
Allows algorithm to reorder elements, copy them, move them, etc. for free.

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Allows algorithm to reorder elements, copy them, move them, etc. for free.

Why is this hard?

- Lower bound needs to hold for *all* algorithms
- How can we simultaneously reason about algorithms as different as mergesort, quicksort, heapsort, ...?

Sorting as Permutations

Think of an array $\mathbf{A}$ as a *permutation*: $\mathbf{A[i]}$ is the $\pi(\mathbf{i})$'th smallest element

$$\mathbf{A} = [\mathbf{23}, \mathbf{14}, \mathbf{2}, \mathbf{5}, \mathbf{76}]$$

Corresponds to $\pi = (\mathbf{3}, \mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{4})$:

$$\pi(\mathbf{0}) = \mathbf{3} \qquad \pi(\mathbf{1}) = \mathbf{2} \qquad \pi(\mathbf{2}) = \mathbf{0} \qquad \pi(\mathbf{3}) = \mathbf{1} \qquad \pi(\mathbf{4}) = \mathbf{4}$$

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Corresponds to $\pi = (3, 2, 0, 1, 4)$:

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Lemma

Given $\mathbf{A}$ with $|\mathbf{A}| = \mathbf{n}$, if can sort in $\mathbf{T(n)}$ comparisons then can find π in $\mathbf{T(n)}$ comparisons

Sorting As Permutations (cont'd)

Lemma

Given $\mathbf{A}$ *with* $|\mathbf{A}| = \mathbf{n}$, *if can sort in* $\mathbf{T(n)}$ *comparisons then can find* π *in* $\mathbf{T(n)}$ *comparisons*

Proof Sketch.

- "Tag" each element of $\mathbf{A}$ with index: $[23, 14, 2, 5, 76] \rightarrow [(23, 0), (14, 1), (2, 2), (5, 3), (76, 4)]$
- Sort tagged $\mathbf{A}$ into tagged $\mathbf{B}$ with $\mathbf{T(n)}$ comparisons: $[(2, 2), (5, 3), (14, 1), (23, 0), (76, 4)]$
- Iterate through to get π: $\pi(2) = 0, \pi(3) = 1, \pi(1) = 2, \pi(0) = 3, \pi(4) = 4$ □

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Corollary

Contrapositive: If need at least $\mathbf{T(n)}$ *comparisons to find* π*, need at least* $\mathbf{T(n)}$ *comparisons to sort!*

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Want to show that it takes $\mathbf{\Omega(n \log n)}$ comparisons to find $\boldsymbol{\pi}$ in comparison model.

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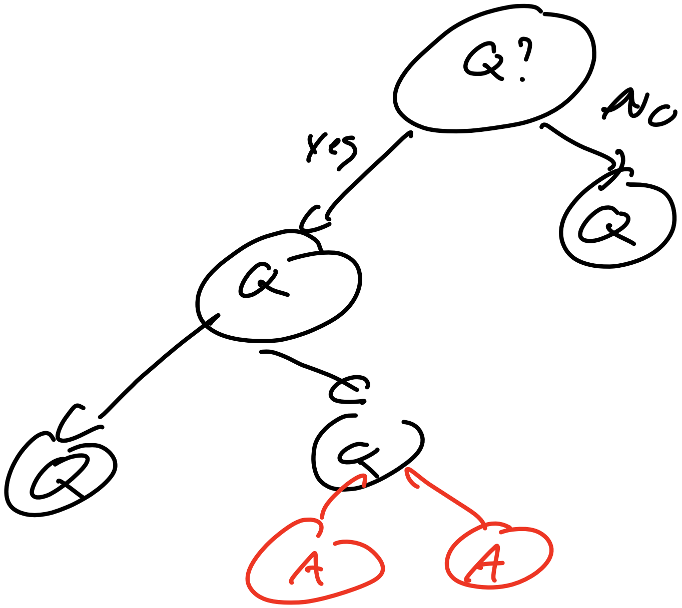
Remind you of anything?

Decision Trees

Model any algorithm as a *binary decision tree*

- Internal nodes: comparisons
- Leaves: permutations



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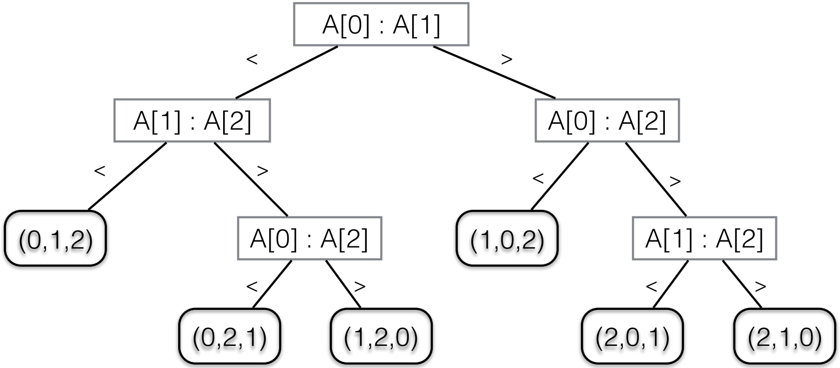

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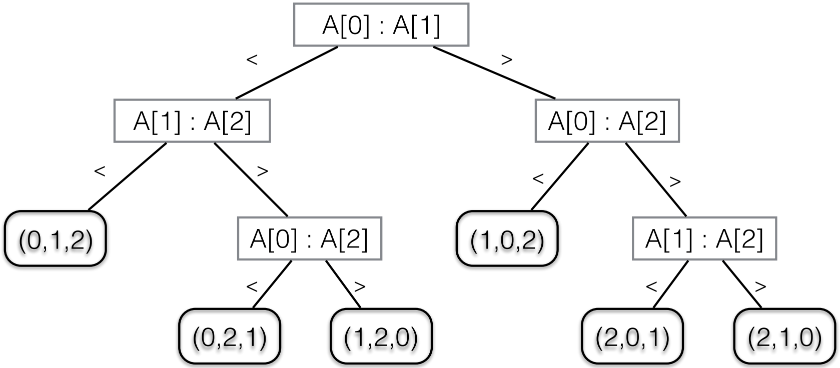

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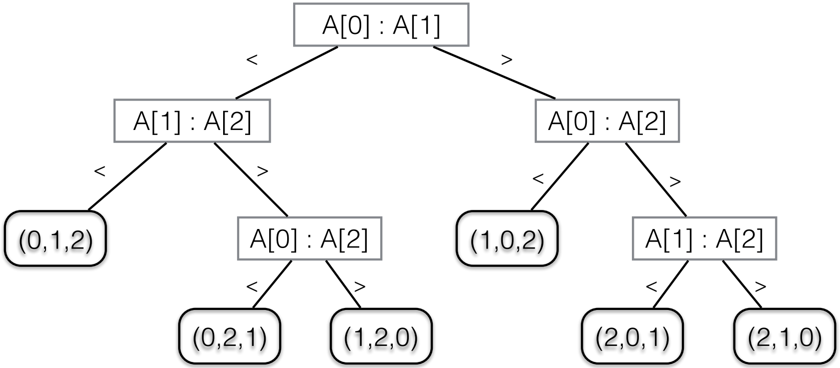
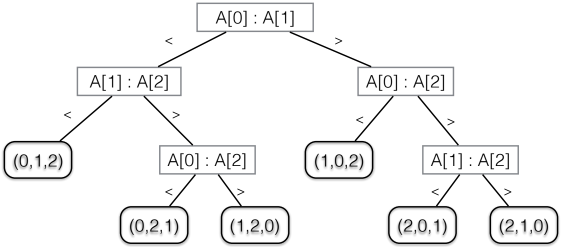


Max # comparisons: **3**

Finishing Up



Scale to general **n**. Consider arbitrary decision tree.

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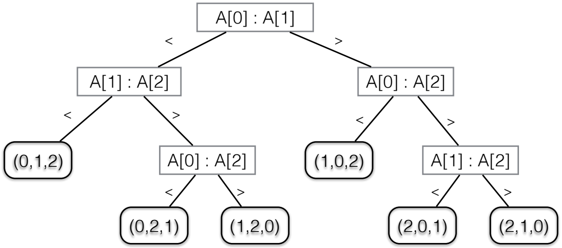

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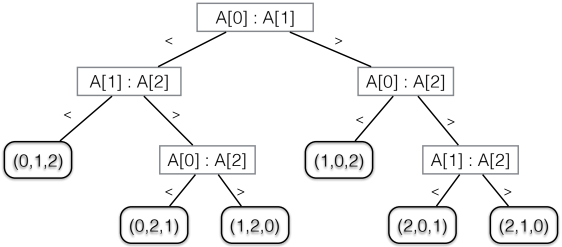

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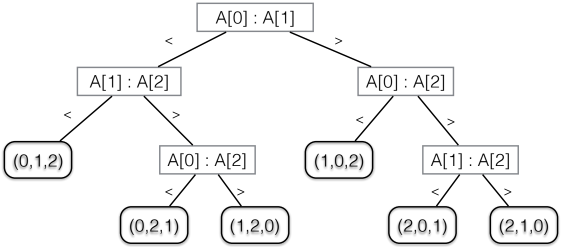

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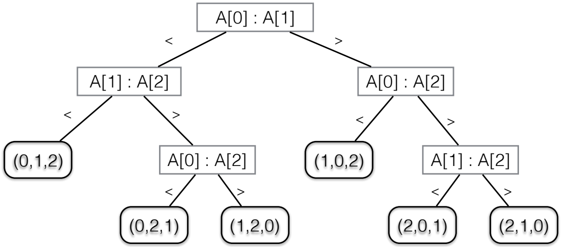

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$$\begin{aligned} \text{Max } \# \text{ comparisons} &= \text{depth of tree} \\ &\geq \mathbf{log_2}(\#\text{ leaves}) \\ &= \mathbf{log_2(n!)} \end{aligned}$$

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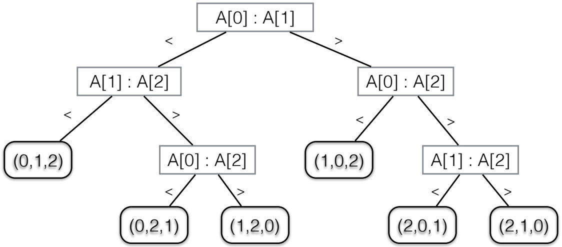

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Sorting Lower Bound Summary

Theorem

Every sorting algorithm in the comparison model must make at least $\log(n!) = \Theta(n \log n)$ *comparisons (in the worst case).*

Proof Sketch.

1. Lower bound on finding permutation $\pi \implies$ lower bound on sorting
2. Any algorithm for finding π is a binary decision tree with $n!$ leaves.
3. Any binary decision tree with $n!$ leaves has depth $\geq \log(n!) = \Theta(n \log n)$

$\implies$ Every algorithm has worst case number of comparisons at least $\Theta(n \log n)$. □

“Linear-Time” Sorting

Bypassing the Lower Bound

What if we're *not* in the comparison model?

- Can do more than just compare elements.

Main example: *integers*.

- What is the $\mathbf{3}$rd bit of $\mathbf{A[0]}$?
- Is $\mathbf{A[0]} \ll \mathbf{k}$ larger than $\mathbf{A[1]} \gg \mathbf{c}$?
- Is $\mathbf{A[0]}$ even?

Same ideas apply to letters, strings, etc.

Counting Sort

Suppose $\mathbf{A}$ consists of $\mathbf{n}$ integers, all in $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{k}-\mathbf{1}\}$.

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Running time: $\mathbf{O}(\mathbf{n}+\mathbf{k})$

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Often want to sort *objects* based on *keys*:

- Each object has a key: integer in $\{0, 1, \ldots, k-1\}$
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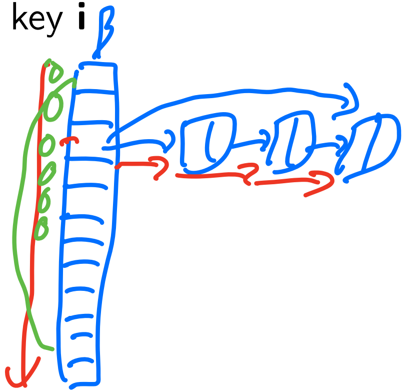

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Stable: if two objects have same key, order between them after sorting is same as before.

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- Assume all numbers have exactly **d** digits (for simplicity: ~~see homework~~).

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If you were sorting cards, with a number on each card, what might you do?

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Divide into $\mathbf{10}$ buckets by first digit, recurse on each bucket by second-digit, etc.

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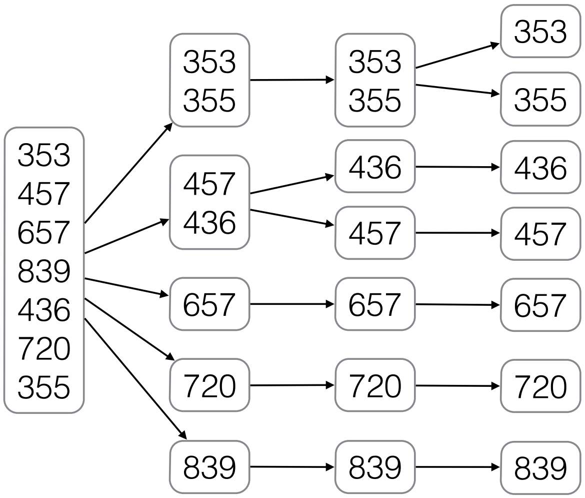

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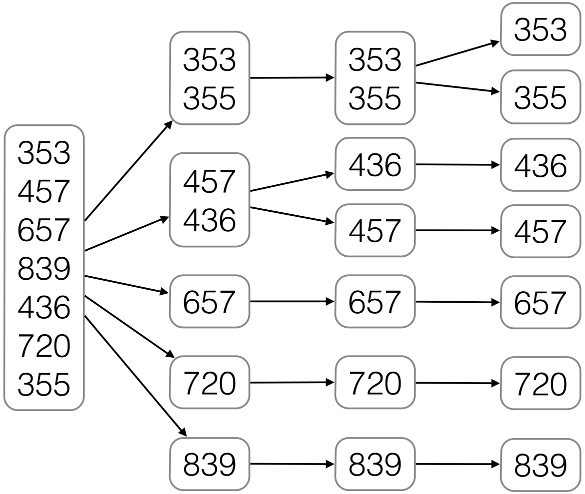

Works, but clunky

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839	→	436	→	353	→	457
436		457		355		657
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Theorem

Radix sort from least significant to most significant is correct if the sort used on each digit is stable.

Least-Significant Radix Sort: Correctness

Proof.

Claim: After $\mathbf{i}$'th iteration, correctly sorted by last $\mathbf{i}$ digits (interpreted as # in $[\mathbf{0}, \mathbf{10^i} - \mathbf{1}]$).

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  - If two numbers have different $\mathbf{i} + \mathbf{1}$ digits, now correct.
  - If two number have same $\mathbf{i} + \mathbf{1}$ digit, were correct and still correct by stability.

□

Least-Significant Radix Sort: Running Time

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Improve to $\mathbf{O(n)}$?

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Fast Radix Sort

Change to go $\mathbf{b}$ digits at a time instead of just $\mathbf{1}$.

- Kind of cheating: look at $\mathbf{b}$ digits in constant time.
- Necessary if we want time better than $\mathbf{nd}$

bucket sorts: $\mathbf{d/b}$
Time per bucket sort: $\mathbf{O(n+k) = O(n+10^b)}$
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Set $\mathbf{b = \log_{10} n}$. If $\mathbf{d = O(\log n)}$, then time

$$\mathbf{O\left(\frac{d}{\log_{10} n}(n+n)\right) = O(n)}$$

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Example: sorting integers between $\mathbf{0}$ and $\mathbf{n^{10}}$. Then $\mathbf{d}$ should be about $\mathbf{\log_{10} n^{10} = 10 \log_{10} n}$, as required.