Lecture 26: Algorithmic Game Theory

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601.433/633 Introduction to Algorithms

Introduction

Algorithmic game theory: (some) intersections of algorithms and game theory (or economics more broadly)

Three subareas:

- Computation of equilibria
- Inefficiency of equilibria
- Algorithmic mechanism design

Today: very fast examples of first two

- See 601.436/636 for a whole class on this!

Two-Player Zero-Sum Games: Penalty Kicks

Penalty kicks in soccer:

- Two players: goalie and kicker
- Too fast to react: both players have to guess.

Model as *matrix game*: matrix $\mathbf{M}$, each entry of form $\mathbf{(a, b)}$

- Kicker picks row and goalie picks column (simultaneously)
- $\mathbf{(a, b)}$: kicker (row player) gets "utility" $\mathbf{a}$, goalie (column player) gets "utility" $\mathbf{b}$
- "Zero-sum": $\mathbf{a + b = 0}$ (so usually just write first value: row player's utility)

Goalie

kicker	Left	Right
Left	(0,0)	(1,-1)
Right	(1, -1)	(0,0)

What should each player do?

Minimax

Two-player zero-sum matrix game: $\mathbf{M} \in \mathbb{R}^{\mathbf{n} \times \mathbf{m}}$, row player tries to maximize, column player tries to minimize.

Natural approach: assume other player knows you well, do as best as possible.

- Row player: choose *distribution* over rows, so that no matter what column player does (even if they know distribution), still get utility
- Penalty kicks:
  - Probability $\mathbf{1/2}$ for each direction. Even if goalie knows, still get utility $\mathbf{1}$ with probability $\mathbf{1/2}$!
  - If we bias at all, then goalie who knows this is more likely to block us: get utility less than $\mathbf{1/2}$ in expectation

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- Choose *minimax* strategy: probability distribution $\mathbf{p}$ over $[\mathbf{n}]$ to maximize

$$\mathbf{V_p} = \min_{\mathbf{j} \in [\mathbf{m}]} \sum_{\mathbf{i} \in [\mathbf{n}]} \mathbf{p_i M_{ij}}$$

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$$\mathbf{V} = \max_{\substack{\text{probability distributions } \mathbf{p} \\ \text{over } [\mathbf{n}]}} \mathbf{V_p} = \max_{\substack{\text{probability distributions } \mathbf{p} \\ \text{over } [\mathbf{n}]}} \min_{\mathbf{j} \in [\mathbf{m}]} \sum_{\mathbf{i} \in [\mathbf{n}]} \mathbf{p_i M_{ij}}$$

Computing Minimax

How to compute minimax strategy?

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How to compute minimax strategy?

$$\max \quad V$$

subject to

$$\sum_{i=1}^{n} p_i = 1$$

$$\sum_{i=1}^{n} p_i M_{ij} \geq V \qquad \forall j \in [m]$$

$0 \leq p_i$ ~~≤ 1~~ $\qquad \forall i \in [n]$

More Penalty Kicks

	Left	Right
Left	(0,0)	(1,-1)
Right	(1, -1)	(0,0)

Kicker (row) minimax:

- **1/2** on each direction
- Guarantees at least **1/2** utility in expectation

Goalie (column) minimax:

- **1/2** on each direction
- Guarantees at least **–1/2** utility in expectation (at most **1/2** loss)

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		1	0
		Left	Right
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		Left	Right
2/3	Left	$(\frac{1}{2}, -\frac{1}{2})$	(1,-1)
1/3	Right	(1, -1)	(0,0)

Kicker (row) minimax:

- $\mathbf{(2/3, 1/3)}$
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Minimax Theorem

Theorem (Minimax Theorem (von Neumann))

Every $\mathbf{2}$*-player zero-sum game has a unique value* $\mathbf{V}$ *such that the minimax strategy for the row player guarantees expected gain of at least* $\mathbf{V}$*, and the minimax strategy for the column player also guarantees expected loss of at most* $\mathbf{V}$*.*

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Proof outside the scope of the course, but not hard.

- Easiest proof: LP duality

General Games and Nash Equilibria

General (one-shot) games: allow more than $\mathbf{2}$ players, utilities don't have to add to $\mathbf{0}$.

- No longer a unique value!

Replace minimax strategies with *Nash equilibria*

- (Randomized) strategy for every player so that no one has incentive to deviate (knowing all other strategies)

Example

Example: two people walking down the sidewalk

	Left	Right
Left	(1,1)	(-1,-1)
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	Left ($\frac{1}{2}$)	Right ($\frac{1}{2}$)
p_L Left	(1,1)	(-1,-1)
p_R Right	(-1, -1)	(1,1)

Nash equilibria:

- Both left
- Both right
- Both $(1/2, 1/2)$
  - Row player: expected utility is 0
  - Suppose deviated to (p_L, p_R) (column player stays at $(1/2, 1/2)$):

$$\frac{1}{2}(1 \cdot p_L - 1 \cdot p_R) + \frac{1}{2}(-1 \cdot p_L + 1 \cdot p_R) = 0$$

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Nash's proof: through Brouwer's fixed-point theorem

- "Every continuous function from a convex compact subset **K** of a Euclidean space to **K** itself has a fixed point."
- Famous and fundamental result in topology
- Non-constructive!

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Question: Can we *compute* Nash equilibria?

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Somewhat tricky to formalize

Attempt 1: Is it **NP**-hard to compute a Nash equilibrium?

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Braess's Paradox

Nash equilibria can behave strangely. Example: *Braess's Paradox* in *routing games*.

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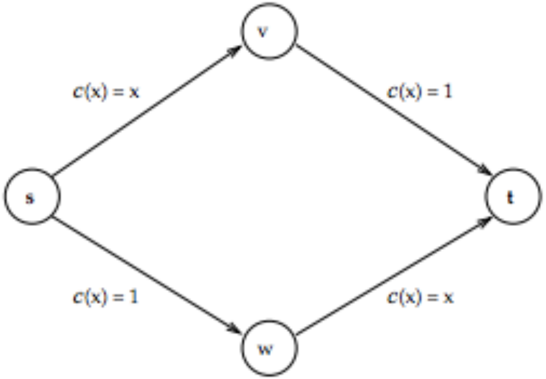

- Huge number of players ($\mathbf{1}/\epsilon$) trying to get from **s** to **t**, each controls ϵ traffic
- Actions: path from **s** to **t**
- Cost of edge: $\mathbf{c(x)}$ where $\mathbf{x}$ is fraction of flow through edge
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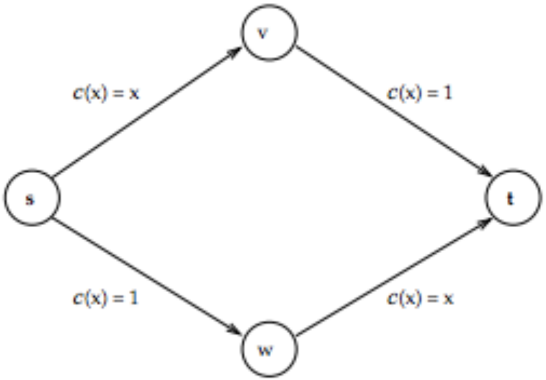

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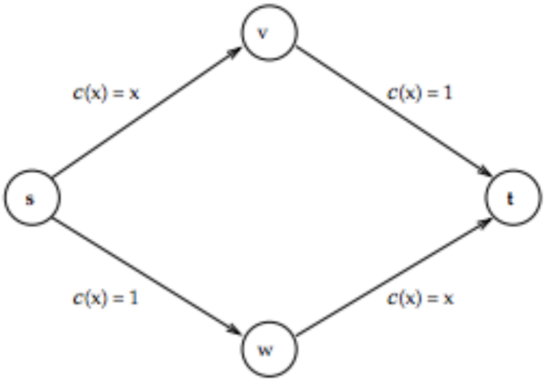

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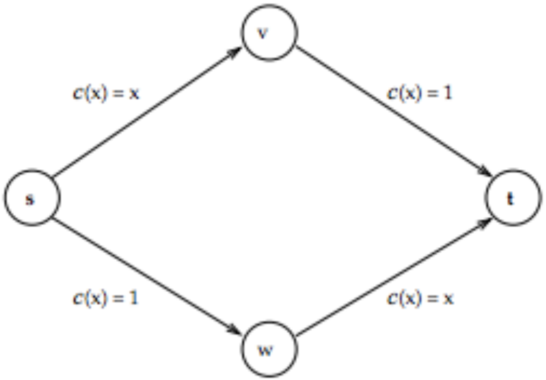

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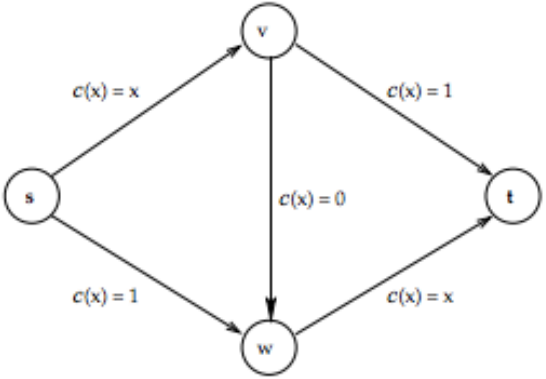

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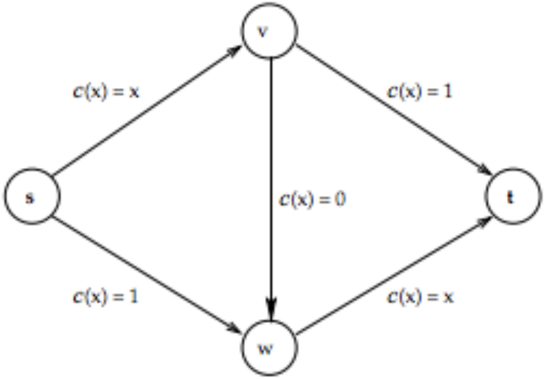

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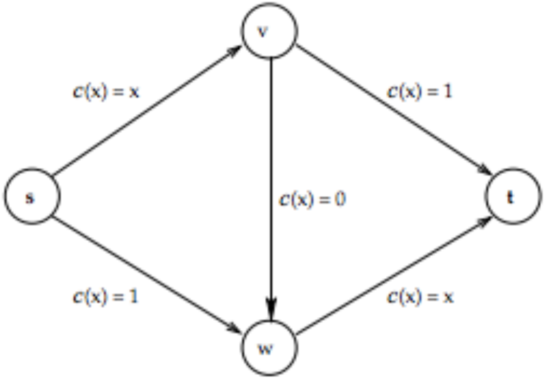

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- So *improved* edges leads to *worse* performance!

Price of Anarchy

Braess's paradox $\implies$ sometime Nash are not "optimal"

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Definition

The *price of anarchy* of a minimization game is $\max_{\mathbf{s} \in \mathcal{S}} \mathbf{W(s)}/\mathbf{OPT}$.

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Routing game example: $\mathbf{OPT} = \mathbf{3/2}$, only one Nash, has cost $\mathbf{2}$.

$\implies$ Price of Anarchy $= \mathbf{2/(3/2)} = \mathbf{4/3}$

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Theorem (Roughgarden)

The price of anarchy in any routing game with linear edge costs is at most **4/3**

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Hope you enjoyed the class, and good luck on the final!