Lecture 23: Approximation Algorithms

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601.433/633 Introduction to Algorithms

Introduction

What should we do if a problem is NP-hard?

- Give up on efficiency?
- Give up on correctness?
- Give up on worst-case analysis?

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Popular answer: *approximation algorithms* (one of my main research areas!)

- Give up on correctness, but in a provable, bounded way.
- Applies to optimization problems only (not pure decision problems)
- Has to run in polynomial time, but can return answer that is *approximately* correct.

Main Definition

Definition

Let $\mathcal{A}$ be some (minimization) problem, and let $\mathbf{I}$ be an instance of that problem. Let $\mathbf{OPT(I)}$ be the cost of the optimal solution on that instance. Let $\mathbf{ALG}$ be a polynomial-time algorithm for $\mathcal{A}$, and let $\mathbf{ALG(I)}$ denote the cost of the solution returned by $\mathbf{ALG}$ on instance $\mathbf{I}$. Then we say that $\mathbf{ALG}$ is an α-*approximation* if

$$\frac{\mathbf{ALG(I)}}{\mathbf{OPT(I)}} \leq \alpha$$

approximation ratio

for all instances $\mathbf{I}$ of $\mathcal{A}$.

- Approximation always at least 1
- For maximization, can either require $\frac{\mathbf{ALG(I)}}{\mathbf{OPT(I)}} \geq \alpha$ (where $\alpha < 1$) or $\frac{\mathbf{OPT(I)}}{\mathbf{ALG(I)}} \leq \alpha$ (where $\alpha > 1$)

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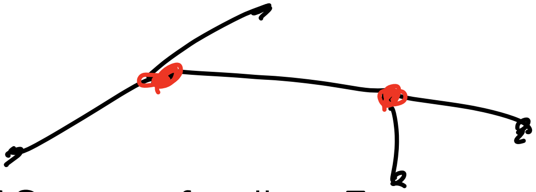
- Also gives "fine-grained" complexity: not all **NP**-hard problems are equally hard!

Vertex Cover



Definition: $\mathbf{S} \subseteq \mathbf{V}$ is a *vertex cover* of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ if $\mathbf{S} \cap \mathbf{e} \neq \emptyset$ for all $\mathbf{e} \in \mathbf{E}$

Definition (VERTEX COVER)

Instance is graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$. Find vertex cover $\mathbf{S}$, minimize $|\mathbf{S}|$.

Last time: VERTEX COVER **NP**-hard (reduction from INDEPENDENT SET)

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So cannot expect to compute a minimum vertex cover efficiently. What about an *approximately* minimum vertex cover?

- Not an approximate vertex cover: still needs to be an actual vertex cover!

Obvious Algorithm 1

$\mathbf{S} = \varnothing$
while there is at least one uncovered edge $\{$
    Pick arbitrary vertex $\mathbf{v}$ incident on at least one uncovered edge
    Add $\mathbf{v}$ to $\mathbf{S}$
$\}$

Obvious Algorithm 1

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while there is at least one uncovered edge {
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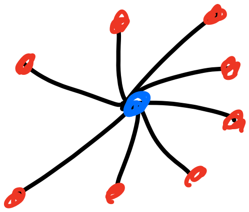
Not a good approximation: star graph.

- **OPT = 1**
- **ALG = n − 1**



Obvious Algorithm 2

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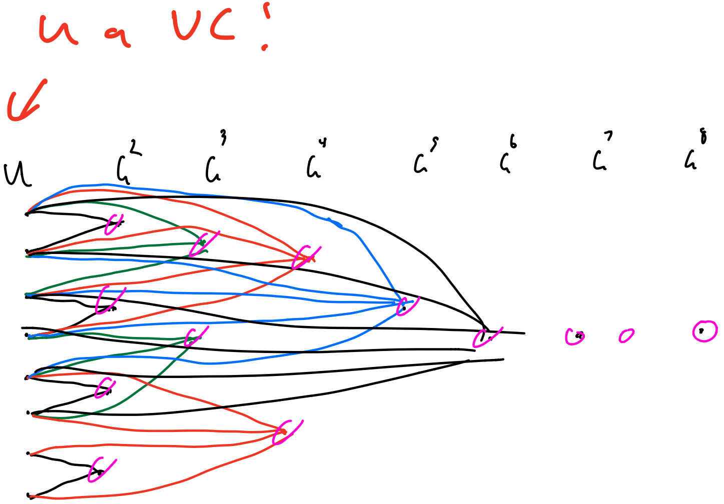
- $|\mathbf{U}| = \mathbf{t}$
- For all $\mathbf{i} \in \{\mathbf{2}, \mathbf{3}, \ldots, \mathbf{t}\}$, divide **U** into $\lfloor \mathbf{t}/\mathbf{i} \rfloor$ disjoint sets of size **i**: $\mathbf{G}^{\mathbf{i}}_{\mathbf{1}}, \mathbf{G}^{\mathbf{i}}_{\mathbf{2}}, \ldots, \mathbf{G}^{\mathbf{i}}_{\lfloor \mathbf{t}/\mathbf{i} \rfloor}$
- Add vertex for each set, edge to all elements.



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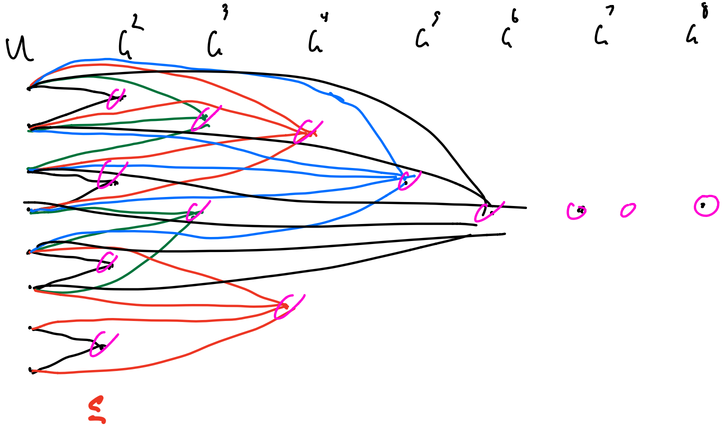

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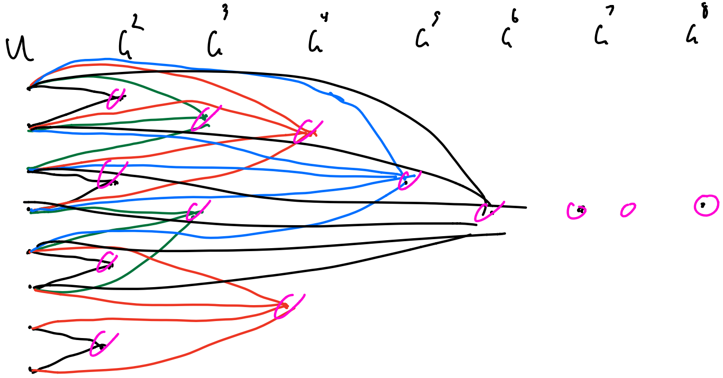

OPT $= \mathbf{t}$

ALG $= \sum_{i=2}^{t} \left\lfloor \frac{t}{i} \right\rfloor \geq \sum_{i=2}^{t} \left(\frac{1}{2} \cdot \frac{t}{i} \right) = \frac{t}{2} \sum_{i=2}^{t} \frac{1}{i} = \Omega(t \log t)$

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Theorem

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Suppose algorithm take $\mathbf{k}$ iterations. Let $\mathbf{L}$ be *edges* chosen by the algorithm, so $|\mathbf{L}| = \mathbf{k}$.

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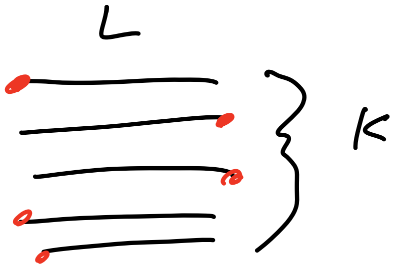
L has structure: it is a matching!



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$\implies \mathbf{ALG}/\mathbf{OPT} \leq 2$.

More Complicated Algorithm: LP Rounding

Write LP for vertex cover:

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$$\begin{array}{lll} \textbf{min} & \sum_{v \in V} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \forall \{u, v\} \in E \\ & 0 \leq x_u \leq 1 & \forall u \in V \end{array}$$

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Question: Is this enough? cost

- Let **OPT(LP)** denote ~~value~~ of optimal LP solution: does **OPT(LP)** = **OPT**?

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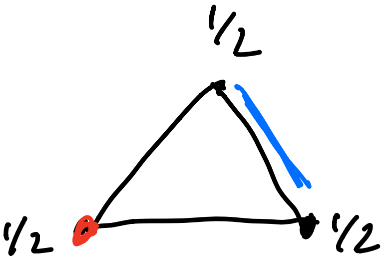

- **OPT** = 2
- **OPT(LP)** = 3/2

LP Structure

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Let $x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$

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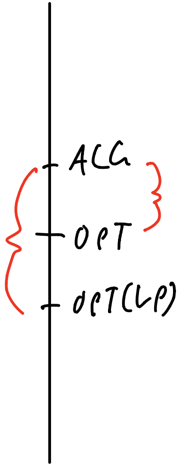

LP Rounding Algorithm

- Solve LP to get $\mathbf{x}^*$ (so $\sum_{\mathbf{v} \in \mathbf{V}} \mathbf{x}^*_{\mathbf{v}} = \mathbf{OPT(LP)}$)
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Polytime: ✓

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$$|S| = \sum_{v\in S} 1 \leq \sum_{v\in S} 2x_v^* \leq 2\sum_{v\in V} x_v^*$$
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$$\sum_{\mathbf{v} \in \mathbf{S}} \mathbf{w}(\mathbf{v}) \leq \sum_{\mathbf{v} \in \mathbf{S}} 2\mathbf{x}^*_\mathbf{v} \mathbf{w}(\mathbf{v}) \leq 2 \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{w}(\mathbf{v}) \mathbf{x}^*_\mathbf{v} = 2 \cdot \mathbf{OPT(LP)} \leq 2 \cdot \mathbf{OPT}$$

Higher level: LP provides *lower bound* on **OPT**. Often main difficulty!

Reductions and Approximation

Proved Vertex Cover **NP**-hard by reduction from Independent Set:

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There is a notion of "approximation-preserving reduction", but it is more involved than a normal reduction.

Max-E3SAT

Recall 3-SAT: CNF formula (AND of ORs) where every clause has $\leq$ **3** literals

- E3-SAT: Same, but every clause has *exactly* three literals (still **NP**-complete)

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Easy *randomized* algorithm: Choose random assignment!

- For each variable $\mathbf{x_i}$, set $\mathbf{x_i} = \mathbf{T}$ with probability $\mathbf{1/2}$ and $\mathbf{F}$ with probability $\mathbf{1/2}$

Max-E3SAT: Analysis

Algorithm: Choose random assignment

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Max-E3SAT: Analysis

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Clause **i**: probability satisfied = **7/8**

Max-E3SAT: Analysis

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Clause **i**: probability satisfied = **7/8**

Random variables:

- For $\mathbf{i} \in \{1, 2, \ldots, m\}$, let $\mathbf{X_i} = \begin{cases} 1 & \text{if clause } \mathbf{i} \text{ satisfied} \\ 0 & \text{otherwise} \end{cases}$
  - $\mathbf{E[X_i] = 7/8}$

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Can be derandomized (method of conditional expectations)

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Theorem (Håstad '01)

Assuming $\mathbf{P} \neq \mathbf{NP}$, *for all constant* $\epsilon > \mathbf{0}$ *there is no polytime* $\left(\frac{\mathbf{7}}{\mathbf{8}} + \epsilon\right)$*-approximation for Max-E3SAT.*