# Lecture 22: NP-Completeness II

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November 11, 2021
601.433/633 Introduction to Algorithms

# Introduction

Last time: Definition of **P**, **NP**, reductions, **NP**-completeness. Proof that Circuit-SAT is **NP**-complete.

Today: more NP-complete problems.

### Definition

A decision problem **Q** is in **NP** (*nondeterministic polynomial time*) if there exists a polynomial time algorithm $\mathbf{V(I,X)}$ (called the *verifier*) such that

1. If **I** is a YES-instance of **Q**, then there is some **X** (usually called the *witness*, *proof*, or *solution*) with size polynomial in $|\mathbf{I}|$ so that $\mathbf{V(I,X)}$ = YES.
2. If **I** is a NO-instance of **Q**, then $\mathbf{V(I,X)}$ = NO for all **X**.

# Reductions

## Definition

A *Many-one* or *Karp* reduction from $\mathbf{A}$ to $\mathbf{B}$ is a function $\mathbf{f}$ which takes arbitrary instances of $\mathbf{A}$ and transforms them into instances of $\mathbf{B}$ so that

1. If $\mathbf{x}$ is a YES-instance of $\mathbf{A}$ then $\mathbf{f(x)}$ is a YES-instance of $\mathbf{B}$.
2. If $\mathbf{x}$ is a NO-instance of $\mathbf{A}$ then $\mathbf{f(x)}$ is a NO-instance $\mathbf{B}$.
3. $\mathbf{f}$ can be computed in polynomial time.

## Definition

Problem $\mathbf{Q}$ is **NP**-*hard* if $\mathbf{Q}' \leq_{\mathbf{p}} \mathbf{Q}$ for all problems $\mathbf{Q}'$ in **NP**. Problem $\mathbf{Q}$ is **NP**-*complete* if it is **NP**-hard and in **NP**.

# Circuit-SAT

## Definition

*Circuit-SAT*: Given a boolean circuit of AND, OR, and NOT gates, with a single output and no loops (some inputs might be hardwired), is there a way of setting the inputs so that the output of the circuit is **1**?

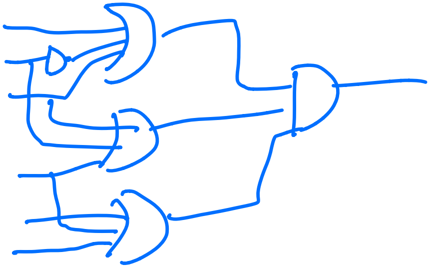
## Theorem

*Circuit-SAT is* **NP***-complete.*

# 3-SAT

Boolean formula:

- Boolean variables $\mathbf{x_1, \ldots, x_n}$
- Literal: variable $\mathbf{x_i}$ or negation $\mathbf{\bar{x}_i}$
- AND: $\wedge$ OR: $\vee$
- $\mathbf{x_1 \vee (\bar{x}_5 \wedge x_7) \wedge (\bar{x}_2 \vee (x_6 \wedge \bar{x}_3)) \ldots}$



Conjunctive normal form (CNF): AND of ORs (clauses)

- $\mathbf{(x_1 \vee \bar{x}_2 \vee \bar{x}_4) \wedge (x_2 \vee x_3) \wedge (x_1 \vee x_4 \vee \bar{x}_6) \ldots}$

### Definition

*3-SAT*: Instance is 3CNF formula $\phi$ (every clause has $\leq$ **3** literals). YES if there is assignment where $\phi$ evaluates to True (satisfying assignment), NO otherwise.

# 3-SAT

**Theorem**

*3-SAT is* **NP***-complete.*

3-SAT in **NP**: witness is assignment, verifier checks that formula evaluates to True on assignment.

3-SAT is **NP**-hard: Show Circuit-SAT $\leq_p$ 3-SAT.

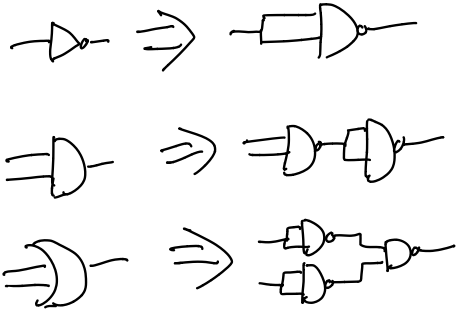
- Don't need to show that $\mathbf{A} \leq_p$ 3-SAT for arbitrary $\mathbf{A} \in \mathbf{NP}$: already know that $\mathbf{A} \leq_p$ Circuit-SAT!

So start with circuit. Want to transform to 3-CNF formula.

# Transformation to NANDs

For simplicity, transform into a circuit with one type of gate: NAND (NOT AND)

- AND/OR/NOT universal, but so is just NAND!



So given circuit $\mathbf{C}$, first transform it into NAND-only circuit.

Input:

- $\mathbf{n}$ "input wires" $\mathbf{x_1, x_2, \ldots, x_n}$
- $\mathbf{m}$ NAND gates: $\mathbf{g_1, \ldots, g_m}$
  - $\mathbf{g_1 = NAND(x_1, x_3)}$, $\mathbf{g_2 = NAND(g_1, x_4)}$, ...
- WLOG, $\mathbf{g_m}$ is the "output gate"

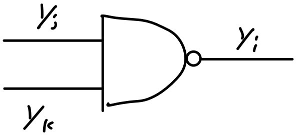
# Reduction to 3-SAT

So given as input a circuit $\mathbf{C}$:

- $\mathbf{n}$ "input wires" $\mathbf{x_1, x_2, \ldots, x_n}$
- $\mathbf{m}$ NAND gates: $\mathbf{g_1, \ldots, g_m}$. Output gate $\mathbf{g_m}$

Need to construct many-one reduction $\mathbf{f}$ to 3-SAT: in polynomial time, construct 3-CNF formula $\mathbf{f(C)}$ such that $\mathbf{f(C)}$ has a satisfying assignment if and only if $\mathbf{C}$ has an input where it outputs $\mathbf{1}$.

**Variables:** $\mathbf{y_1, y_2, \ldots, y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+m}}$ (one for each wire)
**Clauses:** For every NAND gate $\mathbf{y_i = NAND(y_j, y_k)}$, create clauses:




- $\mathbf{y_i \vee y_j \vee y_k}$ (if $\mathbf{y_j = 0}$ and $\mathbf{y_k = 0}$ then $\mathbf{y_i = 1}$)
- $\mathbf{y_i \vee \bar{y}_j \vee y_k}$ (if $\mathbf{y_j = 1}$ and $\mathbf{y_k = 0}$ then $\mathbf{y_i = 1}$)
- $\mathbf{y_i \vee y_j \vee \bar{y}_k}$ (if $\mathbf{y_j = 0}$ and $\mathbf{y_k = 1}$ then $\mathbf{y_i = 1}$)
- $\mathbf{\bar{y}_i \vee \bar{y}_j \vee \bar{y}_k}$ (if $\mathbf{y_j = 1}$ and $\mathbf{y_k = 1}$ then $\mathbf{y_i = 0}$)

Also add clause $\mathbf{(y_{m+n})}$ (want output gate to be $\mathbf{1}$)

# Analysis

**Theorem**

*This is a many-one reduction from Circuit-SAT to 3-SAT.*

Polytime: ✓

Suppose $\mathbf{C}$ YES of Circuit-SAT

$\Longrightarrow$ $\exists$ setting $\mathbf{x}$ of input wires so $\mathbf{g_m} = \mathbf{1}$

$\Longrightarrow$ $\exists$ assignment of $\mathbf{y_1, \ldots y_{m+n}}$ so that all clauses are satisfied:

- $\mathbf{y_i} = \mathbf{x_i}$ if $\mathbf{i} \leq \mathbf{n}$
- $\mathbf{y_i} = \mathbf{g_{i-n}}$ if $\mathbf{i} > \mathbf{n}$

$\Longrightarrow$ $\mathbf{f(C)}$ YES of 3-SAT

Suppose $\mathbf{f(C)}$ YES of 3-SAT

$\Longrightarrow$ $\exists$ assignment $\mathbf{y}$ to variables so that all clauses satisfied

$\Longrightarrow$ $\exists$ setting $\mathbf{x}$ of input wires so $\mathbf{g_m} = \mathbf{1}$:

- $\mathbf{x_i} = \mathbf{y_i}$
- Output of gate $\mathbf{g_i} = \mathbf{y_{i+n}}$ (by construction)
- So $\mathbf{g_m} = \mathbf{1}$ (since $(\mathbf{y_{m+n}})$ is a clause)

$\Longrightarrow$ $\mathbf{C}$ a YES instance of Circuit-SAT

# General Methodology to Prove **Q** **NP**-Complete

1. Show **Q** is in **NP**
   - Can verify witness for YES
   - Can catch false witness for NO (or contrapositive: if witness is verified, then a YES instance)
2. Find some **NP**-hard problem **A**. Reduce *from* **A** *to* **Q**:
   - Given instance **I** of **A**, turn into $\mathbf{f(I)}$ of **Q** (in time polynomial in $|\mathbf{I}|$)
   - **I** YES of **A** if and only if $\mathbf{f(I)}$ YES of **Q**

Notes:

- Careful about direction of reduction!!!!
- Need to handle *arbitrary* instances of **A**, but can turn into very structured instances of **Q**
- Often easiest to prove NO direction via contrapositive, to turn into statement about YES:
  - **I** YES of **A** $\implies$ $\mathbf{f(I)}$ YES of **Q**
  - $\mathbf{f(I)}$ YES of **Q** $\implies$ **I** YES of **A**
  - So proving "both directions", but reduction only in one direction.

# Clique

**Definition:** A *clique* in an undirected graph $\mathbf{G = (V, E)}$ is a set $\mathbf{S \subseteq V}$ such that $\{\mathbf{u, v}\} \in \mathbf{E}$ for all $\mathbf{u, v} \in \mathbf{S}$

**Definition (Clique)**

Instance is a graph $\mathbf{G = (V, E)}$ and an integer $\mathbf{k}$. YES if $\mathbf{G}$ contains a clique of size at least $\mathbf{k}$, NO otherwise.

**Theorem**

Clique *is* **NP***-complete.*

In **NP**:

- Witness: $\mathbf{S \subseteq V}$
- Verifier: Checks if $\mathbf{S}$ is a clique and $|\mathbf{S}| \geq \mathbf{k}$
  - If $(\mathbf{G, k})$ a YES instance: there is a clique $\mathbf{S}$ of size $\geq \mathbf{k}$ on which verifier returns YES
  - If $(\mathbf{G, k})$ a NO instance: $\mathbf{S}$ cannot be clique of size $\geq \mathbf{k}$, so verifier always returns NO

# CLIQUE is **NP**-hard

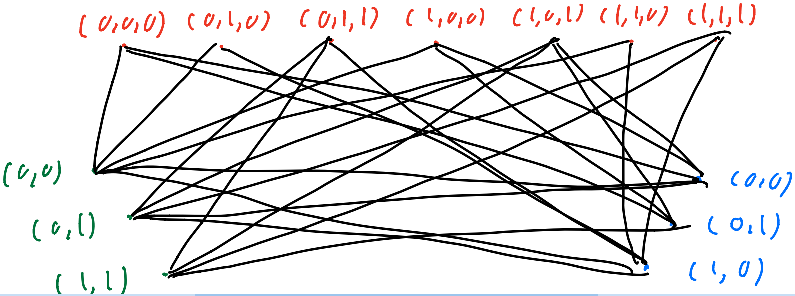
Prove by reducing 3-SAT to CLIQUE

- For arbitrary $\mathbf{A} \in \mathbf{NP}$, would have $\mathbf{A} \leq_p$ Circuit-SAT $\leq_p$ 3-SAT $\leq_p$ CLIQUE

Given 3-SAT formula **F** (with **n** variables and **m** clauses), set $\mathbf{k} = \mathbf{m}$ and create graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$:

- For every clause of **F**, for every satisfying assignment to the clause, create vertex
- Add an edge between consistent assignments

**Example:** $\mathbf{F} = (x_1 \vee x_2 \vee \bar{x}_4) \wedge (\bar{x}_3 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3)$



# 3-SAT to CLIQUE reduction analysis

Polytime: ✓

If $\mathbf{F}$ YES of 3-SAT:

- There is some satisfying assignment $\mathbf{x}$
- For every clause, choose vertex corresponding to $\mathbf{x}$. Let $\mathbf{S}$ be chosen vertices
- $|\mathbf{S}| = \mathbf{m} = \mathbf{k}$, and clique since all consistent (since all from $\mathbf{x}$)

$\implies$ $(\mathbf{G}, \mathbf{k})$ YES of CLIQUE

If $(\mathbf{G}, \mathbf{k})$ YES of CLIQUE:

- There is some clique $\mathbf{S}$ of size $\mathbf{k} = \mathbf{m}$
- Must contain exactly one vertex from each clause (since clique of size $\mathbf{m}$)
- Since clique, all assignments consistent $\implies$ there is an assignment that satisfies all clauses

$\implies$ $\mathbf{F}$ YES of 3-SAT

# Independent Set

**Definition:** $\mathbf{S} \subseteq \mathbf{V}$ is an *independent set* in $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ if $\{\mathbf{u}, \mathbf{v}\} \notin \mathbf{E}$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{S}$

**Definition (Independent Set)**

Instance is graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and integer $\mathbf{k}$. YES if $\mathbf{G}$ has an independent set of size $\geq \mathbf{k}$, NO otherwise.

**Theorem**

Independent Set *is* **NP***-complete.*

In **NP**:

- Witness is $\mathbf{S} \subseteq \mathbf{V}$. Verifier checks that $|\mathbf{S}| \geq \mathbf{k}$ and no edges in $\mathbf{S}$
- If $(\mathbf{G}, \mathbf{k})$ a YES instance then such an $\mathbf{S}$ exists $\implies$ verifier returns YES on it.
- If $(\mathbf{G}, \mathbf{k})$ a NO then verifier will return NO on every $\mathbf{S}$.

# Independent Set is NP-hard

Reduce from: Clique

- Given instance $(G, k)$ of Clique, create "complement graph" $H$: same vertex set, with $\{u, v\} \in E(H)$ if and only if $\{u, v\} \notin E(G)$
- Instance $(H, k)$ of Independent Set

If $(G, k)$ YES of Clique:

$\Longrightarrow$ Clique $S \subseteq V$ of $G$ with $|S| \geq k$

$\Longrightarrow$ $S$ an independent set in $H$

If $(H, k)$ YES of Independent Set:

$\Longrightarrow$ Independent set $S \subseteq V$ in $H$ with $|S| \geq k$

$\Longrightarrow$ $S$ a clique in $G$

# Vertex Cover

**Definition:** $\mathbf{S} \subseteq \mathbf{V}$ is a *vertex cover* of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ if $\mathbf{S} \cap \mathbf{e} \neq \varnothing$ for all $\mathbf{e} \in \mathbf{E}$

**Definition (Vertex Cover)**

Instance is graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, integer $\mathbf{k}$. YES if $\mathbf{G}$ has a vertex cover of size $\leq \mathbf{k}$, NO otherwise.

**Theorem**

Vertex Cover *is* **NP**-*complete*

In **NP**:

- Witness is $\mathbf{S} \subseteq \mathbf{V}$. Verifier checks that $|\mathbf{S}| \leq \mathbf{k}$ and every edge has at least one endpoint in $\mathbf{S}$
- If $(\mathbf{G}, \mathbf{k})$ a YES instance then such an $\mathbf{S}$ exists $\implies$ verifier returns YES on it.
- If $(\mathbf{G}, \mathbf{k})$ a NO then verifier will return NO on every $\mathbf{S}$.

# Vertex Cover is NP-hard

Reduce from Independent Set

- Given instance $(G = (V, E), k)$ of Independent Set, create instance $(G, n - k)$ of Vertex Cover (where $n = |V|$)

If $(G, k)$ a YES instance of Independent Set:

$\implies$ $G$ has an independent set $S$ with $|S| \geq k$

$\implies$ $V \setminus S$ a vertex cover of $G$ of size $\leq n - k$

$\implies$ $(G, n - k)$ a YES instance of Vertex Cover

If $(G, n - k)$ a YES instance of Vertex Cover:

$\implies$ $G$ has a vertex cover $S$ of size at most $n - k$

$\implies$ $V \setminus S$ an independent set of $G$ of size at least $k$

$\implies$ $(G, k)$ a YES instance of Independent Set