Lecture 20: Linear Programming

Michael Dinitz

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601.433/633 Introduction to Algorithms

Introduction

Today: What, why, and juste a taste of how

- Entire course on linear programming over in AMS. Super important topic!
- Fast algorithms in theory and in practice.

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- Entire course on linear programming over in AMS. Super important topic!
- Fast algorithms in theory and in practice.

Why: Even more general than max-flow, can still be solved in polynomial time!

- Max flow important in its own right, but also because it can be used to solve many other things (max bipartite matching)
- Linear programming: important in its own right, but also even more general than max-flow.
- Can model many, many problems!

Example: Planning Your Week (pre-COVID)

168 hours in a week. How much time to spend:

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- Partying (**P**)
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Constraints:

- $\mathbf{E} \geq \mathbf{56}$ (at least 8 hours/day sleep, shower, etc.)
- $\mathbf{P} + \mathbf{E} \geq \mathbf{70}$ (need to stay sane)
- $\mathbf{S} \geq \mathbf{60}$ (to pass your classes)
- $\mathbf{2S} + \mathbf{E} - \mathbf{3P} \geq \mathbf{150}$ (too much partying requires studying or sleep)

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Question: Suppose "happiness" is $\mathbf{2P} + \mathbf{3E}$. Can we find a feasible solution maximizing this?

Linear Programming

Input (a "linear program"):

- $\mathbf{n}$ variables $\mathbf{x_1, \ldots, x_n}$ (take values in $\mathbb{R}$)
- $\mathbf{m}$ *non-strict linear inequalities* in these variables (constraints)
  - E.g.: $\mathbf{3x_1 + 4x_2 \leq 6}$, $\quad \mathbf{0 \leq x_1 \leq 3}$ $\quad \mathbf{x_2 - 3x_3 + 2x_7 = 17}$
  - Not allowed (examples): $\mathbf{x_2 x_3 \geq 5}$, $\quad \mathbf{x_4 < 2}$, $\quad \mathbf{x_5 + \log x_2 \geq 4}$
- Possibly a *linear* objective function
  - $\mathbf{\max 2x_3 - 4x_5}$, $\quad \mathbf{\min \frac{5}{2}x_4 + x_2}$, $\quad \ldots$

Goals:

- Feasibility: Find values for $\mathbf{x}$'s that satisfy all constraints
- Optimization: Find feasible solutions maximizing/minimizing objective function

Both achievable in polynomial time, reasonably fast!

Planning your week as an LP

Variables: $\mathbf{P}, \mathbf{E}, \mathbf{S}$

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$$\mathbf{max} \quad \mathbf{2P + E}$$

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$$\begin{aligned} \textbf{max} \quad & \mathbf{2P + E} \\ \text{subject to} \quad & \mathbf{E \geq 56} \\ & \mathbf{S \geq 60} \\ & \mathbf{2S + E - 3P \geq 150} \\ & \mathbf{P + E \geq 70} \end{aligned}$$

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& \mathbf{P \geq 0} \\
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\end{aligned}
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When using an LP to model your problem, need to be sure that *all* aspects of your problem included!

Operations Research-style Example

Four different manufacturing plants for making cars:

	labor	materials	pollution
Plant 1	2	3	15
Plant 2	3	4	10
Plant 3	4	5	9
Plant 4	5	6	7

Operations Research-style Example

Four different manufacturing plants for making cars:

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- Need to produce at least **400** cars at plant 3 (labor agreement) $x_3 \geq 400$
- Have **3300** total hours of labor, **4000** units of material
- Environmental law: produce at most **12000** pollution
- Make as many cars as possible

OR example as an LP

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Variables: $\mathbf{x_i}$ = # cars produced at plant $\mathbf{i}$, for $\mathbf{i} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$

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Objective: $\max \; \mathbf{x_1 + x_2 + x_3 + x_4}$

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$$x_3 \geq 400$$

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Constraints:

$$x_3 \geq 400$$

$$2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 3300$$

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$$\mathbf{2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 3300}$$

$$\mathbf{15x_1 + 10x_2 + 9x_3 + 7x_4 \leq 12000}$$

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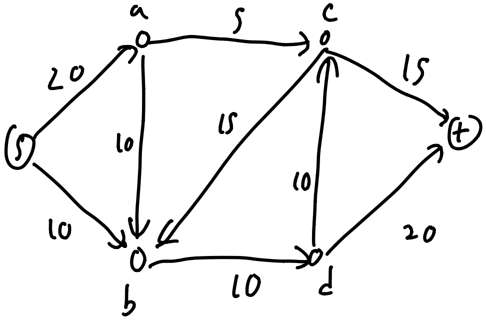
Constraints:

$$
\begin{aligned}
x_3 &\geq 400 \\
2x_1 + 3x_2 + 4x_3 + 5x_4 &\leq 3300 \\
15x_1 + 10x_2 + 9x_3 + 7x_4 &\leq 12000 \\
x_i &\geq 0 \qquad \forall i \in \{1, 2, 3, 4\}
\end{aligned}
$$

$3x_1 + 4x_2 + 5x_3 + 6x_4 \leq 4000$

Max Flow as LP



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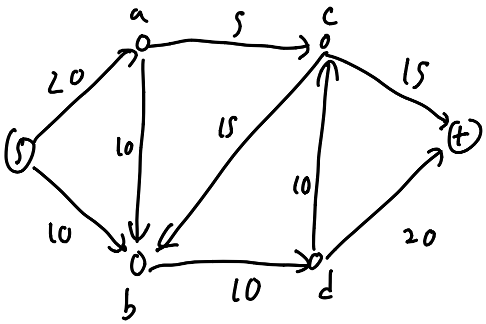

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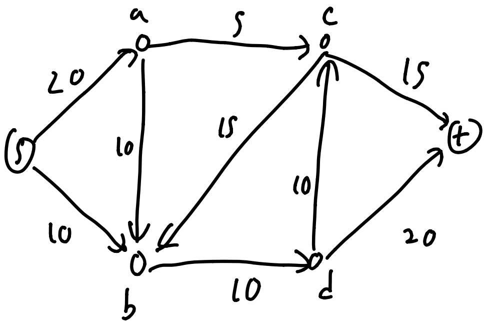

Variables: $\mathbf{f(e)}$ for all $\mathbf{e} \in \mathbf{E}$

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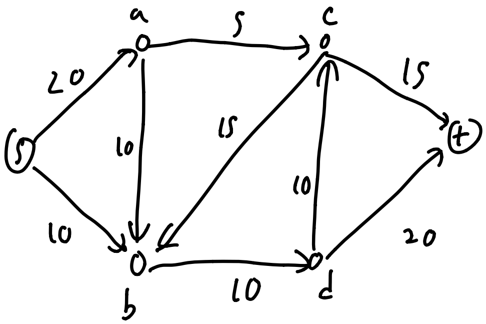

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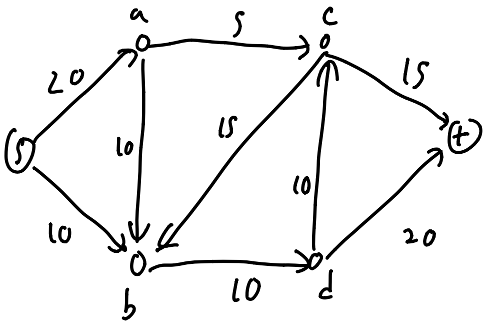

Variables: $\mathbf{f(e)}$ for all $\mathbf{e} \in \mathbf{E}$

Objective: $\max \sum_v \left(f(s,v) - \sum_v f(v,s)\right)$

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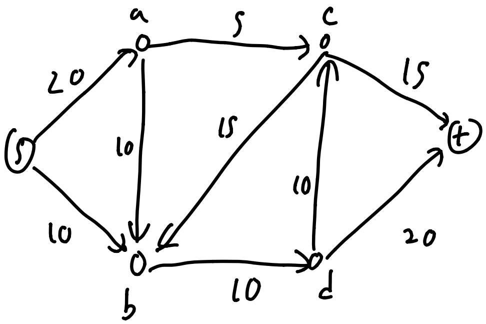

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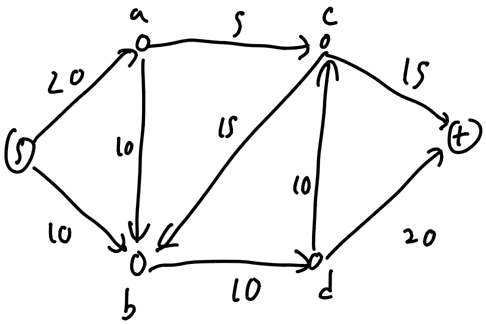

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Objective: $\max \sum_v f(s,v) - \sum_v f(v,s)$

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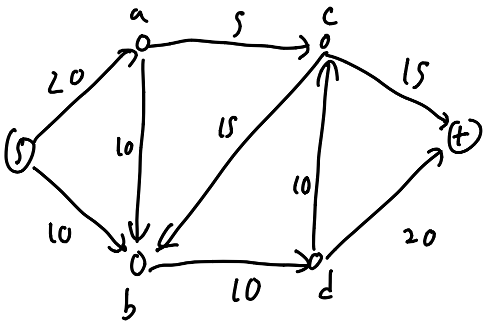

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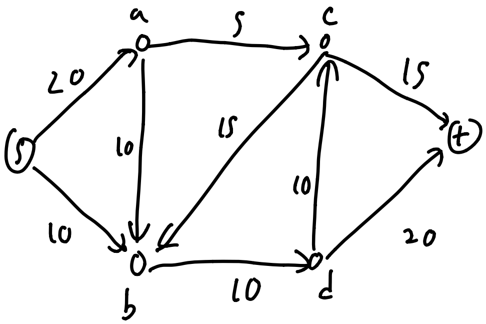

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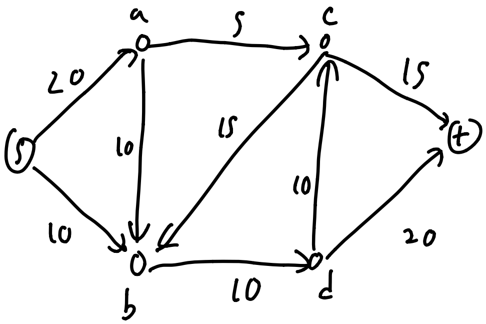

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So can solve max-flow and min-cut (slower) by using generic LP solver

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Generalization of max-flow with multiple commodities that can't mix, but use up same capacity

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Generalization of max-flow with multiple commodities that can't mix, but use up same capacity

Setup:

- Directed graph $G = (V, E)$
- Capacities $c : E \to \mathbb{R}_{\geq 0}$
- k source-sink pairs $\{(s_i, t_i)\}_{i \in [k]}$

Goal: send flow of commodity i from s_i to t_i, max total flow sent across all commodities

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Variables: $\mathbf{f_i(e)}$ for all $\mathbf{e} \in \mathbf{E}$ and for all $\mathbf{i} \in [\mathbf{k}]$. Flow of commodity $\mathbf{i}$ on edge $\mathbf{e}$

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Objective: $\max \sum_{i=1}^{k} \left(\sum_v (f_i(s_i, v) - \sum_v f_i(v, s_i)) \right)$

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Multicommodity Flow

$\{1, 2, \ldots, k\}$

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$$\sum_{i=1}^{k} f_i(e) \leq c(e) \qquad \forall e \in E$$

$$f_i(e) \geq 0 \qquad \forall e \in E,\ \forall i \in [k]$$

Concurrent Flow

Multicommodity flow, but:

- Also given *demands* $\mathbf{d} : [\mathbf{k}] \to \mathbb{R}_{\geq 0}$
- Question: Is there a multicommodity flow that sends at least $\mathbf{d(i)}$ commodity-$\mathbf{i}$ flow from $\mathbf{s_i}$ to $\mathbf{t_i}$ for all $\mathbf{i} \in [\mathbf{k}]$?

Concurrent Flow

Variables: $\mathbf{f_i(e)}$ for all $\mathbf{e} \in \mathbf{E}$ and for all $\mathbf{i} \in [\mathbf{k}]$.

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Variables: $\mathbf{f_i}(\mathbf{e})$ for all $\mathbf{e} \in \mathbf{E}$ and for all $\mathbf{i} \in [\mathbf{k}]$.

Constraints:

Concurrent Flow

Multicommodity flow, but:

- Also given *demands* $\mathbf{d} : [\mathbf{k}] \to \mathbb{R}_{\geq 0}$
- Question: Is there a multicommodity flow that sends at least $\mathbf{d(i)}$ commodity-$\mathbf{i}$ flow from $\mathbf{s_i}$ to $\mathbf{t_i}$ for all $\mathbf{i} \in [\mathbf{k}]$?

Variables: $\mathbf{f_i(e)}$ for all $\mathbf{e} \in \mathbf{E}$ and for all $\mathbf{i} \in [\mathbf{k}]$.

Constraints:

$$\sum_{v} f_i(v,u) - \sum_{v} f_i(u,v) = 0 \qquad \forall i \in [k],\ \forall u \in V \setminus \{s_i, t_i\}$$

$$\sum_{i=1}^{k} f_i(e) \leq c(e) \qquad \forall e \in E$$

$$f_i(e) \geq 0 \qquad \forall e \in E,\ \forall i \in [k]$$

Concurrent Flow

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$$\sum_{v} \left(f_i(s_i,v) - \sum_{v} f_i(v,s_i) \right) \geq d(i) \qquad \forall i \in [k]$$

Maximum Concurrent Flow

If answer is no: how much do we need to scale down demands so that there is a multicommodity flow?

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- λ

Objective: $\max \lambda$

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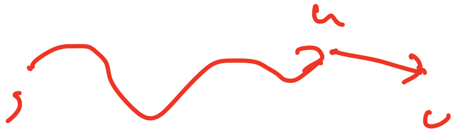
$$f_i(e) \ge 0 \qquad \forall e \in E,\ \forall i \in [k]$$

$$\sum_{v} f_i(s_i,v) - \sum_{v} f_i(v,s_i) \ge \lambda d(i) \qquad \forall i \in [k]$$

Shortest $\mathbf{s}-\mathbf{t}$ path

Very surprising LP!
Variables: $\mathbf{d_v}$ for all $\mathbf{v} \in \mathbf{V}$: shortest-path distance from $\mathbf{s}$ to $\mathbf{v}$

$$\begin{aligned} \textbf{max} \quad & \mathbf{d_t} \\ \text{subject to} \quad & \mathbf{d_s} = \mathbf{0} \\ & \mathbf{d_v} \leq \mathbf{d_u} + \ell(\mathbf{u}, \mathbf{v}) \qquad \forall (\mathbf{u}, \mathbf{v}) \in \mathbf{E} \end{aligned}$$



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Correctness Theorem: Let $\vec{\mathbf{d}}^*$ denote the optimal LP solution. Then $\mathbf{d}^*_\mathbf{t} = \mathbf{d}(\mathbf{s}, \mathbf{t})$

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$\le$: Let $\mathbf{P = (s = v_0, v_1, \ldots, v_k = t)}$ be shortest $\mathbf{s \to t}$ path.
Prove by induction: $\mathbf{d_{v_i}^* \le d(s, v_i)}$ for all $\mathbf{i}$

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Base case: $i = 0$ ✓

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Inductive step: $\mathbf{d^*_{v_i} \leq d^*_{v_{i-1}} + \ell(v_{i-1}, v_i) \leq d(s, v_{i-1}) + \ell(v_{i-1}, v_i) = d(s, v_i)}$

LP constraint induction shortest path

Algorithms for LPs

Geometry

To get intuition: think of LPs *geometrically*

- Space: $\mathbb{R}^{\mathbf{n}}$ (one dimension per variable
- Linear constraint: halfspace (one side of a hyperplane)
- Feasible region: intersection of halfspaces. *Convex Polytope* (usually just called a *polytope*)

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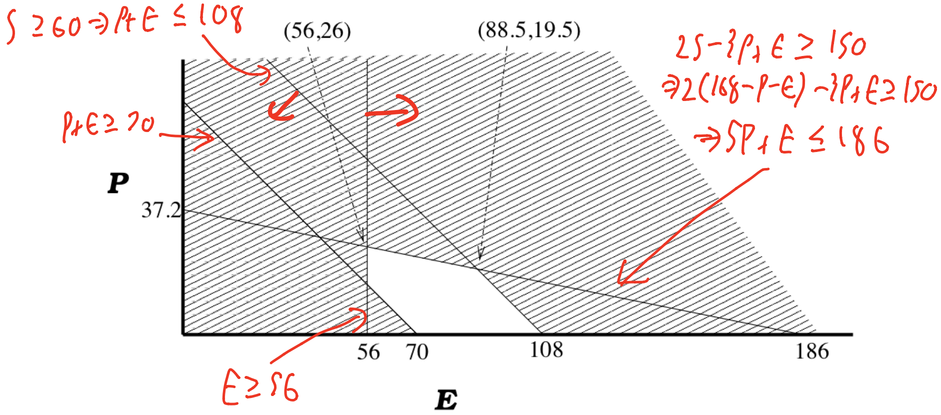
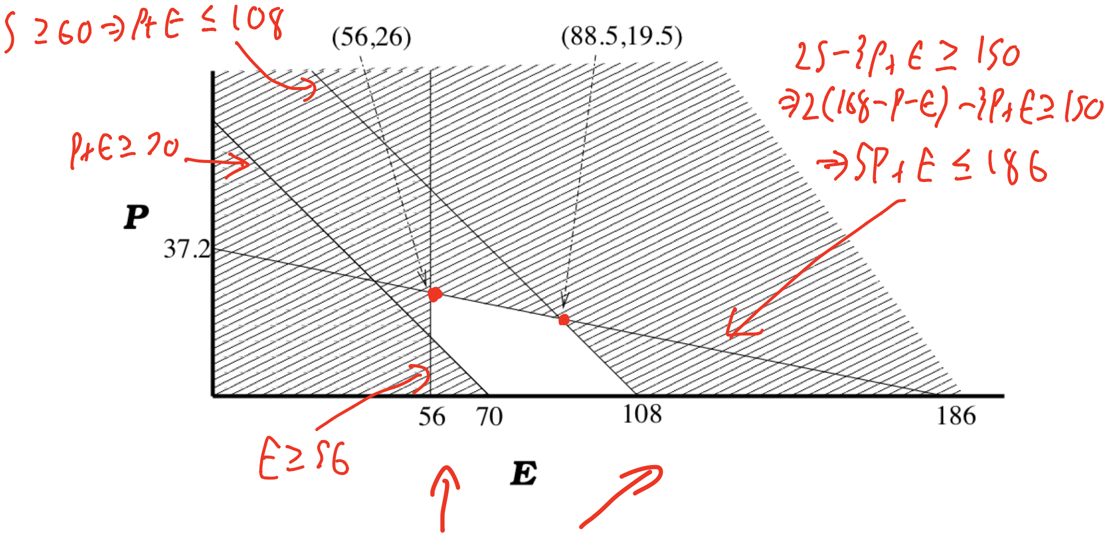
Example: planning your week

- **3** variables $\mathbf{S}, \mathbf{P}, \mathbf{E}$ so $\mathbb{R}^3$
- But $\mathbf{S} + \mathbf{P} + \mathbf{E} = \mathbf{168} \implies \mathbf{S} = \mathbf{168} - \mathbf{P} - \mathbf{E}$
- Make this substitution, get $\mathbb{R}^2$



Geometry (cont'd)



Objective: feasible solution "furthest" along specified direction

- **max P: (56, 26)**
- **max 2P + E: (88.5, 19.5)**

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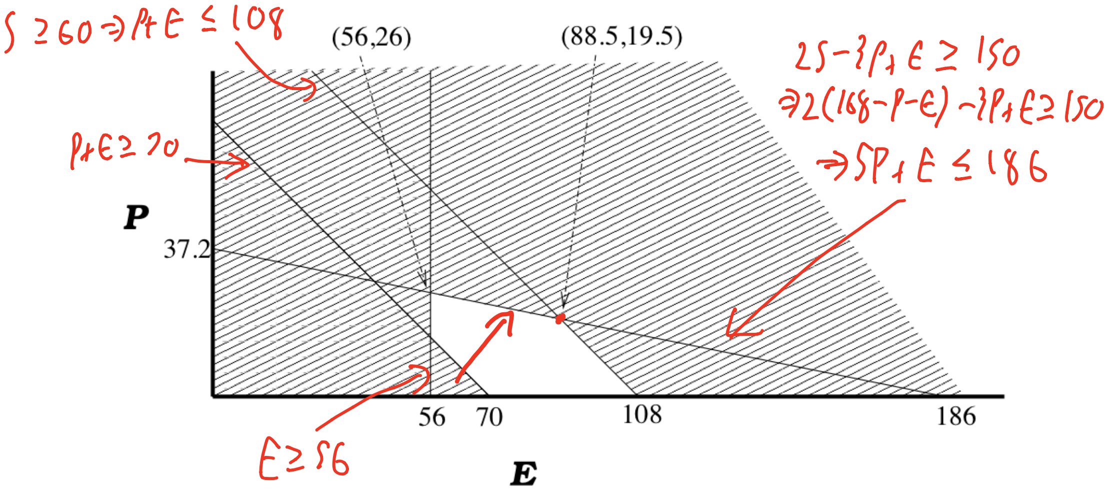

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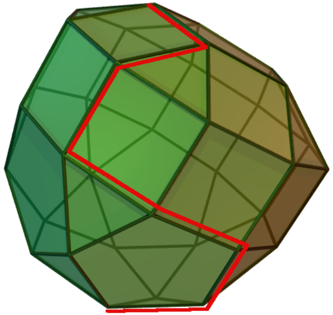
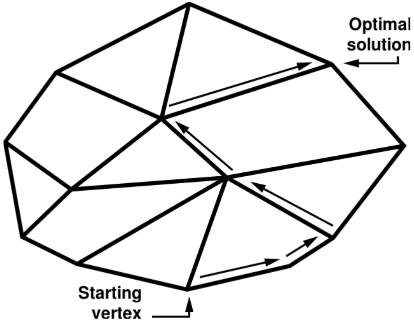
Main theorem: optimal solution is always at a "corner" (also called a "vertex")

Simplex Algorithm [Dantzig 1940's]

Initialize $\vec{\mathbf{x}}$ to an arbitrary corner
while(a neighboring corner $\vec{\mathbf{x}}'$ of $\vec{\mathbf{x}}$ has better objective value) {
    $\vec{\mathbf{x}} \leftarrow \vec{\mathbf{x}}'$
}
return $\vec{\mathbf{x}}$





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Theorem: Simplex returns the optimal solution.

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- Some theory to explain discrepancy ("smoothed analysis")

Ellipsoid Algorithm [Khachiyan 1980]

First polytime algorithm!

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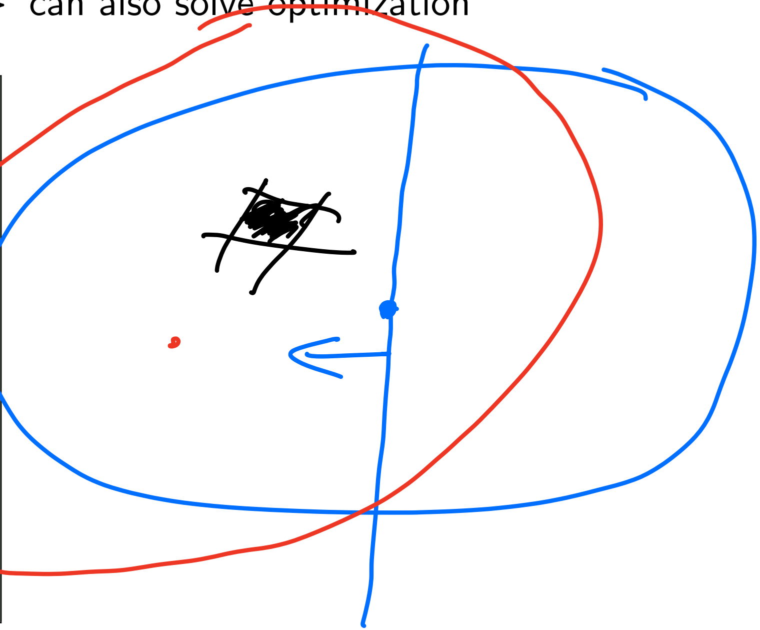
First polytime algorithm!

Designed to just solve feasibility question $\implies$ can also solve optimization

- Start with ellipsoid **E** containing feasible region **P** (if it exists)
- Let **x** be center of **E**
- While(**x** not feasible)
  - Find a hyperplane **H** through **x** such that all of **P** on one side
  - Let $\mathbf{E}'$ be the half-ellipsoid of **E** defined by **H**
  - Find a new ellipsoid $\hat{\mathbf{E}}$ containing $\mathbf{E}'$ so that $\mathbf{vol}(\hat{\mathbf{E}}) \leq \left(1 - \frac{1}{\mathbf{n}}\right) \mathbf{vol}(\mathbf{E})$
  - Let $\mathbf{E} = \hat{\mathbf{E}}$ and let **x** be center of $\hat{\mathbf{E}}$



Analysis

Extremely complicated!

Geometry of ellipsoids: can always find an ellipsoid containing a half-ellipsoid with at most $(1 - 1/n)$ of the volume of the original

- Using inequality from last time: after n iterations, volume drops by $\left(1 - \frac{1}{n}\right)^n \leq 1/e$ factor
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In practice: horrible.

Interior Point Methods (Karkmarkar's Algorithm)

Fast in both theory and practice!