Lecture 19: Max-Flow II

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601.433/633 Introduction to Algorithms

Introduction

Last time:

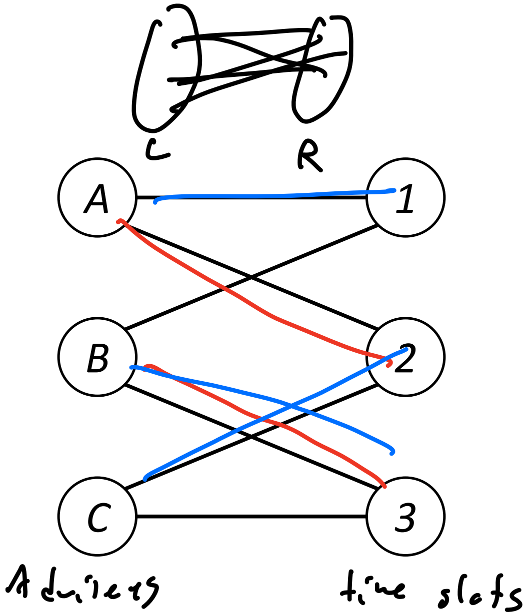
- Max-Flow = Min-Cut
- Can compute max flow and min cut using Ford-Fulkerson: while residual graph has an $\mathbf{s} \to \mathbf{t}$ path, push flow along it.
  - Corollary: if all capacities integers, max-flow is integral
  - If max-flow has value $\mathbf{F}$, time $\mathbf{O(F(m+n))}$ (if all capacities integers)
  - Exponential time!

Today:

- Important setting where FF is enough: max bipartite matching
- Two ways of making FF faster: Edmonds-Karp

Max Bipartite Matching

Setup



Definition

A graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is *bipartite* if $\mathbf{V}$ can be partitioned into two parts $\mathbf{L}, \mathbf{R}$ such that every edge in $\mathbf{E}$ has one endpoint in $\mathbf{L}$ and one endpoint in $\mathbf{R}$.

Definition

A *matching* is a subset $\mathbf{M} \subseteq \mathbf{E}$ such that $\mathbf{e} \cap \mathbf{e}' = \emptyset$ for all $\mathbf{e}, \mathbf{e}' \in \mathbf{M}$ with $\mathbf{e} \neq \mathbf{e}'$ (no two edges share an endpoint)

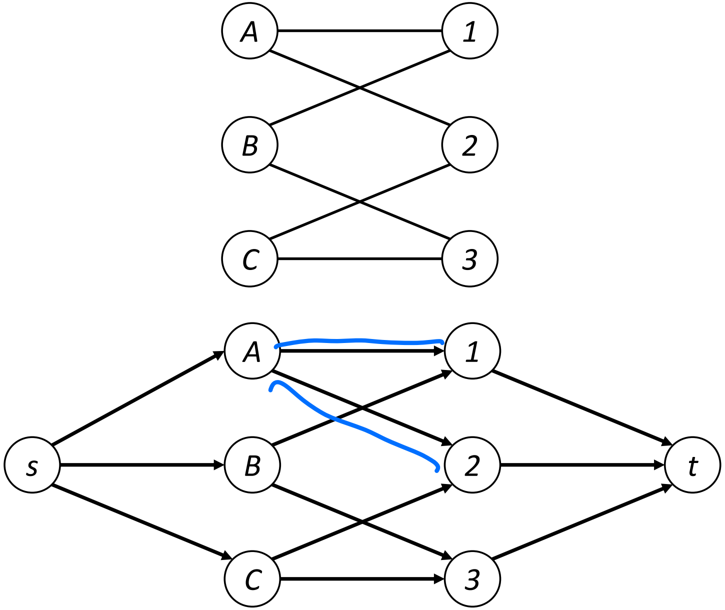
Bipartite Maximum Matching: Given bipartite graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, find matching $\mathbf{M}$ maximizing $|\mathbf{M}|$

- Extremely important problem, doesn't seem to have much to do with flow!

Algorithm

Give all edges capacity **1**
Direct all edges from **L** to **R**
Add source **s** and sink **t**
Add edges of capacity **1** from **s** to **L**
Add edges of capacity **1** from **R** to **t**

Run FF to get flow **f**
Return $\mathbf{M = \{e \in L \times R : f(e) > 0\}}$



Correctness

Claim: M is a matching

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Proof: capacities in $\{0,1\} \implies f(e) \in \{0,1\}$ for all $\mathbf{e}$ (integrality)

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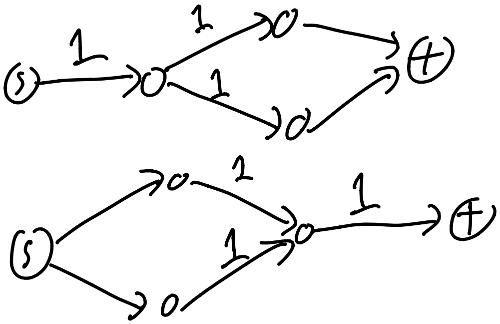

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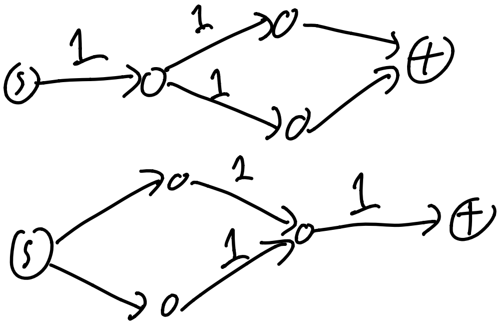

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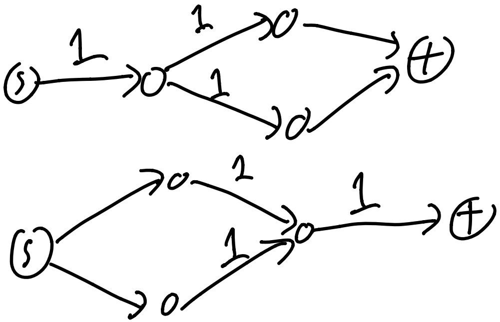

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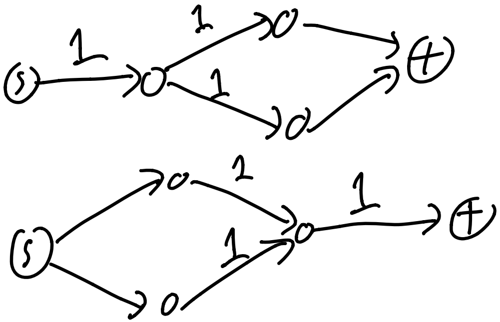

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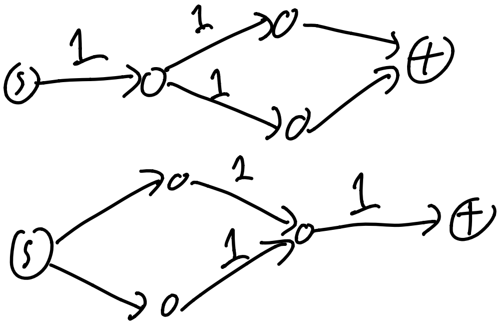

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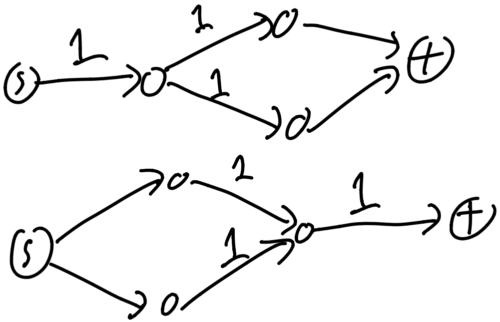

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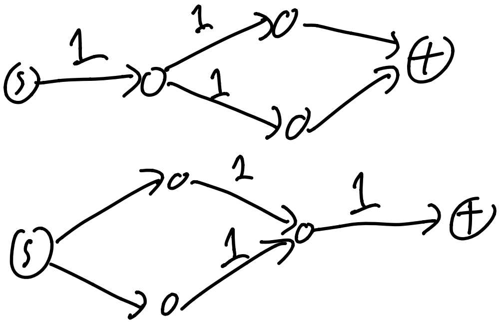

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- Contradiction

Running Time

Running Time:

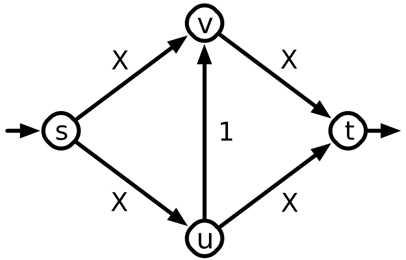
- $\mathbf{O(n+m)}$ to make new graph
- $|\mathbf{f}| = |\mathbf{M}| \leq \mathbf{n/2}$ iterations of FF

$\Longrightarrow$ $\mathbf{O(n(m+n)) = O(mn)}$ time (assuming $\mathbf{m \geq \Omega(n)}$)

Edmonds-Karp

Intuition

Bad example for Ford-Fulkerson:



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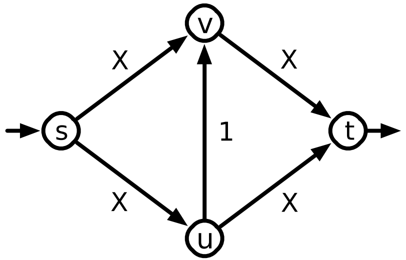

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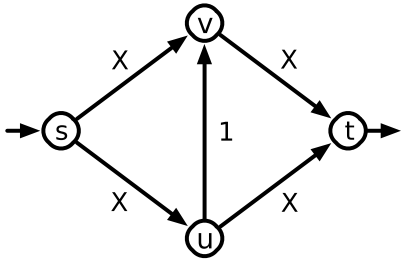

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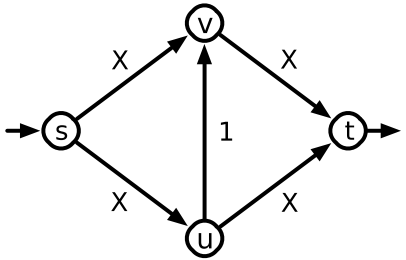

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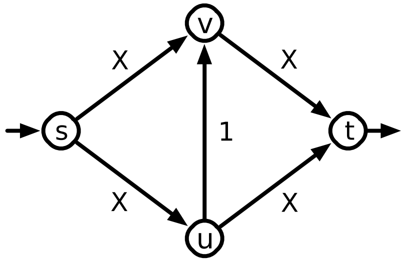

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- "Widest" path: push as much flow as possible each iteration

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- Correct, since FF. Running time?

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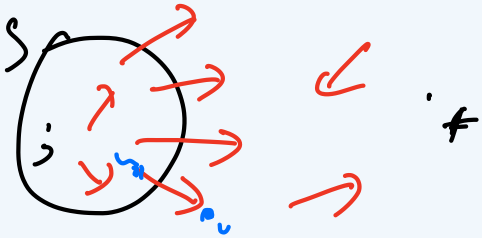

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Does this implies at most $\mathbf{m}$ iterations?

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Theorem

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$\Longrightarrow$ If $\mathbf{i > m \ln F}$, amount remaining to be sent at most

$$\mathbf{F(1 - 1/m)^i < F(1 - 1/m)^{m \ln F} \leq F(e^{-1/m})^{m \ln F} = F \cdot e^{-\ln F} = 1}$$

But all capacities integers, so must be finished!

Finishing EK1

Modified version of Dijkstra: find widest path in $\mathbf{O(m \log n)}$ time

- Total time $\mathbf{O(m \log n \cdot m \log F) = O(m^2 \log n \log F)}$
- Polynomial time!

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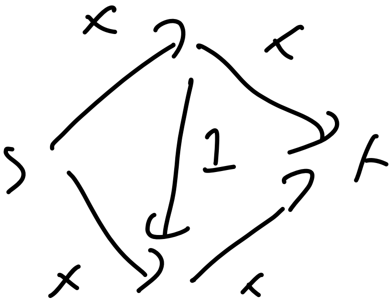

Question: can we get running time independent of **F**?

- *Strongly* polynomial-time algorithm.

Edmonds-Karp #2

Again use Ford-Fulkerson, but pick *shortest* augmenting path (unweighted)

- Ignore capacities, just find augmenting path with fewest hops!
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Theorem

EK2 has at most $\mathbf{O(mn)}$ *iterations, so at most* $\mathbf{O(m^2n)}$ *running time (if* $\mathbf{m \geq n}$*)*

Proof (sketch) of EK2

(in residual graph)

Idea: prove that distance from **s** to **t** (unweighted) goes up by at least one every ≤ **m** iterations.

Proof (sketch) of EK2

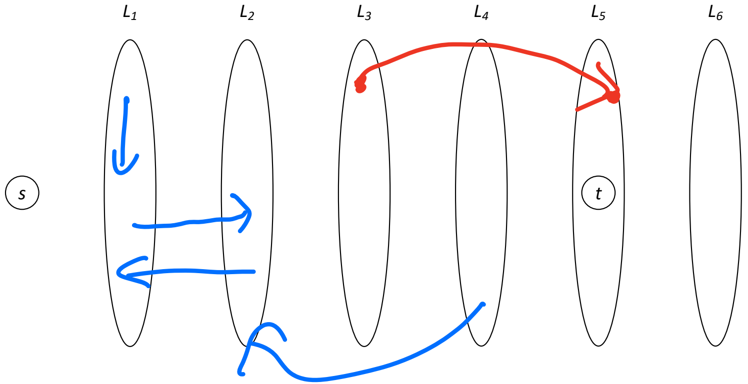
Idea: prove that distance from $\mathbf{s}$ to $\mathbf{t}$ (unweighted) goes up by at least one every $\leq \mathbf{m}$ iterations.

- Distance initially $\geq \mathbf{1} \implies$ distance $> \mathbf{n}$ after at most $\mathbf{mn}$ iterations
- Only distance larger than $\mathbf{n}$ is ∞: no $\mathbf{s} \rightarrow \mathbf{t}$ path

$\implies$ Terminates after at most $\mathbf{mn}$ iterations.

Proof (sketch) of EK2 (continued)

Suppose **s** → **t** distance is **d**.
"Lay out" residual graph in levels by BFS (distance from **s**)



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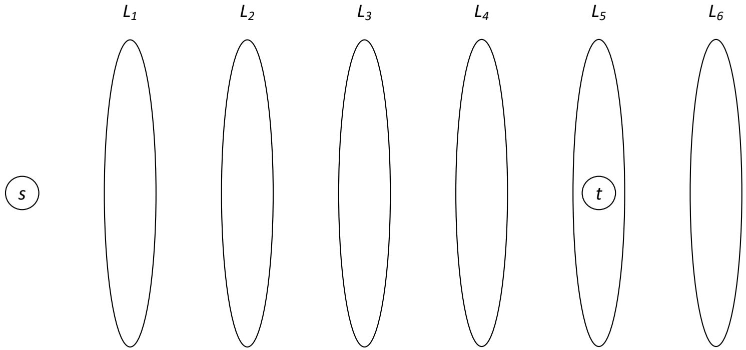

Edge types:

- Forward edges: **1** level
- Edges inside level
- Backwards edges

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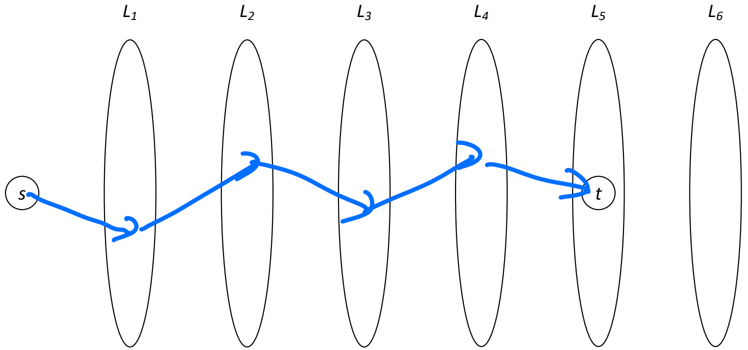
"Lay out" residual graph in levels by BFS (distance from **s**)



Edge types:

- Forward edges: **1** level
- Edges inside level
- Backwards edges

What happens when we choose a *shortest* augmenting path?

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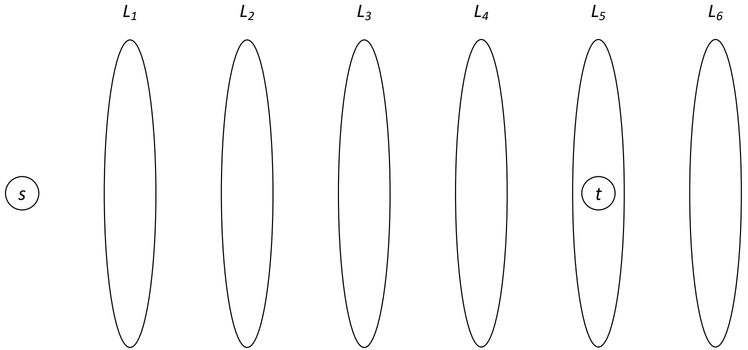

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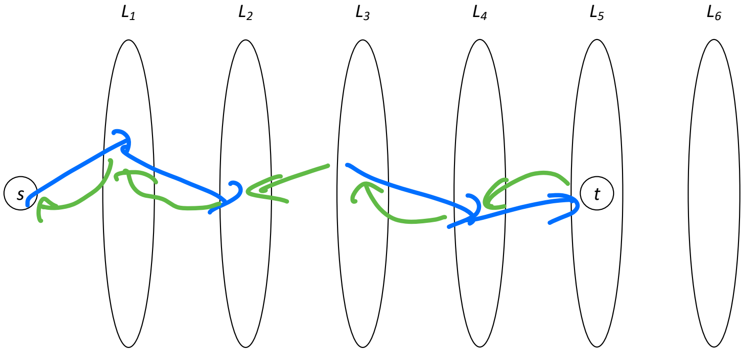

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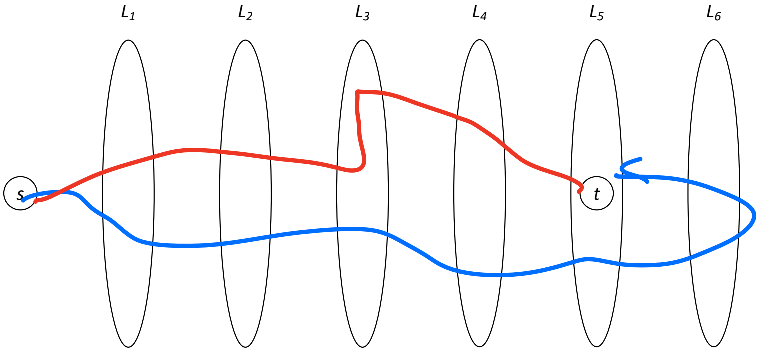

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So after **m** iterations (same layout): no path using only forward edges $\implies$ distance larger than **d**!

Finishing EK2

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Finishing EK2

So at most $\mathbf{mn}$ iterations. Each iteration unweighted shortest path: BFS, time $\mathbf{O(m+n)}$

Total time: $\mathbf{O(mn(m+n)) = O(m^2n)}$. Independent of $\mathbf{F}$!

Extensions

Many better algorithms for max-flow: *blocking flows* (Dinitz's algorithm (not me)), *push-relabel* algorithms, etc.

- CLRS has a few of these.
- State of the art:
  - Strongly polynomial: $\mathbf{O(mn)}$. Orlin [2013] & King, Rao, Tarjan [1994]
  - Weakly Polynomial: $\tilde{\mathbf{O}}(\mathbf{m}^{\frac{3}{2}-\frac{1}{328}} \log \mathbf{U})$ (where $\mathbf{U}$ is maximum capacity). Gao, Liu, Peng [2021]

Many other variants of flows, some of which are just $\mathbf{s-t}$ max flow in disguise!

- Min-Cost Max-Flow: every edge also has a cost. Find minimum cost max-flow. Can be solved with just normal max flow!