Lecture 17: Matroids and the Greedy Algorithm

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October 26, 2021

601.433/633 Introduction to Algorithms

Introduction

Last time: somewhat greedy algorithm (Prim's), extremely greedy algorithm (Kruskal's)

Question: when does greedy algorithm return optimal solution?

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Weighted Set System:

- Universe $\mathbf{U}$
- Collection $\mathcal{I} \subseteq \mathbf{2^U}$ (so $\mathbf{I} \subseteq \mathbf{U}$ for all $\mathbf{I} \in \mathcal{I}$). Called *independent sets*
- Weights $\mathbf{w} : \mathbf{U} \to \mathbb{R}^+$

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Problem: find *max weight* independent set

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MST: weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{w})$. Find MST.

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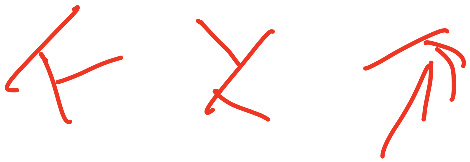
- $\mathcal{I} = \{\mathbf{F} \subseteq \mathbf{E} : (\mathbf{V}, \mathbf{F})$ a forest$\}$



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For any spanning tree $\mathbf{T}$:

$$\mathbf{w}'(\mathbf{T}) = \sum_{\mathbf{e} \in \mathbf{T}} \mathbf{w}'(\mathbf{e}) = \sum_{\mathbf{e} \in \mathbf{T}} (\bar{\mathbf{w}} - \mathbf{w}(\mathbf{e})) = (\mathbf{n} - \mathbf{1})\bar{\mathbf{w}} - \sum_{\mathbf{e} \in \mathbf{T}} \mathbf{w}(\mathbf{e}) = (n-1)\bar{w} - w(T)$$

$$= \sum_{e \in T} \bar{w} - \sum_{e \in F} w(e)$$

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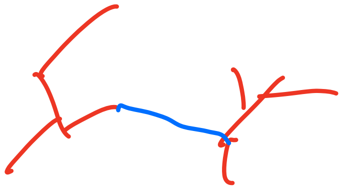

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For any tree $\mathbf{T}$:

$$\mathbf{w'(T) = \sum_{e \in T} w'(e) = \sum_{e \in T} (\bar{w} - w(e)) = (n-1)\bar{w} - \sum_{e \in T} w(e)}$$

So under weights $\mathbf{w'}$, max-weight IS $=$ max-weight forest $=$ max-weight tree $=$ min-weight tree (weights $\mathbf{w}$)

- So finding max-weight forest $=$ finding min spanning tree.

Useful Properties of Forests

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Proof Sketch that Forests have Augmentation Property.

Suppose false: no edge in $\mathbf{F_2} \setminus \mathbf{F_1}$ can be added to $\mathbf{F_1}$. Let $\mathbf{c_1} = \#$ components in $\mathbf{F_1}$, $\mathbf{c_2} = \#$ components in $\mathbf{F_2}$

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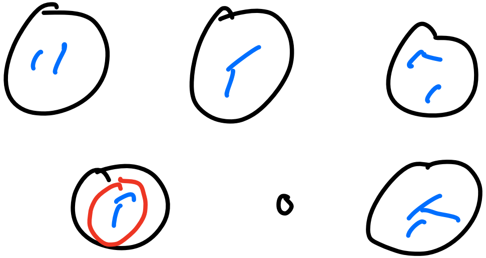

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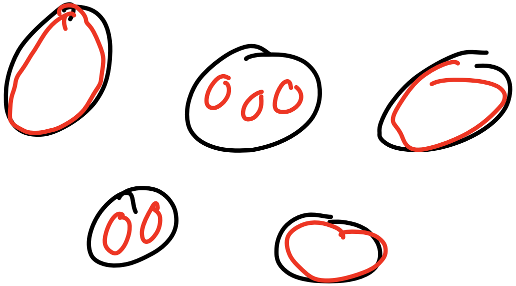

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Contradiction. □

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$(\mathbf{U}, \mathcal{I})$ is a *matroid* if the following three properties hold:

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Warmup: In any matroid, the maximal independent sets (called bases) have the same size (called the rank of the matroid).

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Matroids: generalize both graph theory and linear algebra!

- Originally invented by Whitney as an attempt to generalize the concept of "linear independence"

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We'll assume we have independence oracle.

Greedy Algorithm

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$\mathbf{F} = \varnothing$
Sort $\mathbf{U}$ by weight (largest to smallest)
For each $\mathbf{u} \in \mathbf{U}$ in sorted order $\{$
    If $\mathbf{F} \cup \{\mathbf{u}\} \in \mathcal{I}$, add $\mathbf{u}$ to $\mathbf{F}$
$\}$
Return $\mathbf{F}$

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Claim: $\mathbf{w(f_i) \geq w(e_i)}$ for all $\mathbf{i}$.

Proof: Suppose false, let $\mathbf{j}$ smallest integer such that $\mathbf{w(f_j) < w(e_j)}$.

Let $\mathbf{F_1} = \{\mathbf{f_1, \ldots, f_{j-1}}\}$ and let $\mathbf{F_2} = \{\mathbf{e_1, \ldots, e_j}\}$

$|\mathbf{F_2}| > |\mathbf{F_1}|$, so by augmentation there is some $\mathbf{e_z} \in \mathbf{F_2} \setminus \mathbf{F_1}$ such that $\mathbf{F_1} \cup \{\mathbf{e_z}\} \in \mathcal{I}$.

$$\mathbf{w(e_z) \geq w(e_j) > w(f_j)}$$

Contradiction! Greedy would add $\mathbf{e_z}$ next, not $\mathbf{f_j}$.

Converse

So greedy works on matroids. Amazing fact: if greedy works, set system is a matroid!

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Theorem

Let $(\mathbf{U}, \mathcal{I})$ be an hereditary set system. If for every weighting $\mathbf{w} : \mathbf{U} \to \mathbb{R}_{\geq 0}$ the greedy algorithm returns a maximum weight independent set, then $(\mathbf{U}, \mathcal{I})$ is a matroid.

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So for hereditary set systems, matroids exactly characterize when the greedy algorithm works!

Proof

Contradiction. Suppose false $\implies$ $(\mathbf{U}, \mathcal{I})$ hereditary but not matroid.

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$\implies \exists \mathbf{F_1}, \mathbf{F_2} \in \mathcal{I}$ such that $|\mathbf{F_1}| < |\mathbf{F_2}|$ but $\mathbf{F_1} \cup \{\mathbf{e}\} \notin \mathcal{I}$ for all $\mathbf{e} \in \mathbf{F_2} \setminus \mathbf{F_1}$

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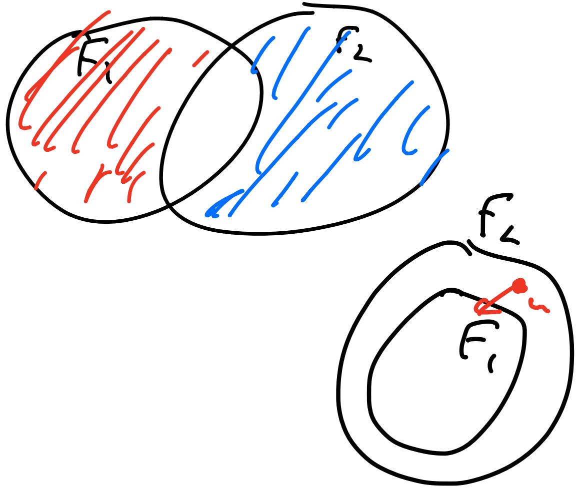

Easy facts:

1. $|\mathbf{F_2} \setminus \mathbf{F_1}| > |\mathbf{F_1} \setminus \mathbf{F_2}|$
2. $|\mathbf{F_2} \setminus \mathbf{F_1}| \geq \mathbf{1}$
3. $|\mathbf{F_1} \setminus \mathbf{F_2}| \geq \mathbf{1}$ (hereditary)

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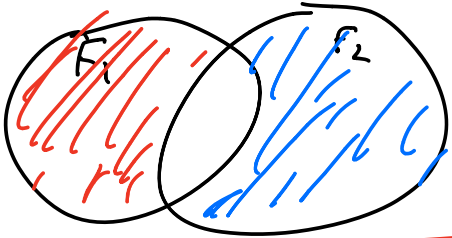

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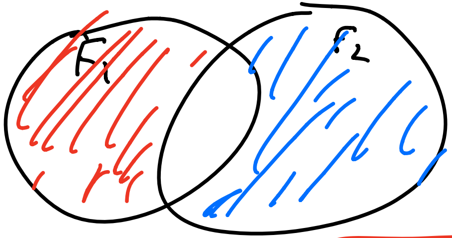

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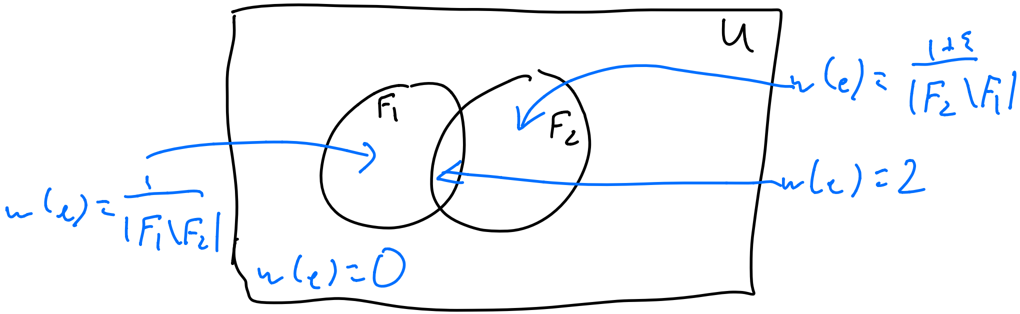
$$\implies \frac{\mathbf{1}}{|\mathbf{F_1} \setminus \mathbf{F_2}|} > \frac{\mathbf{1} + \epsilon}{|\mathbf{F_2} \setminus \mathbf{F_1}|}$$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$\mathbf{w}(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|}$$

$$= 2|F_1 \cap F_2| + 1$$

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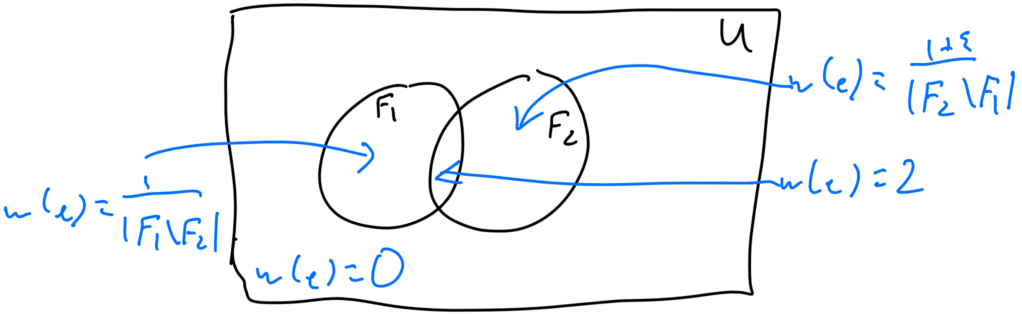

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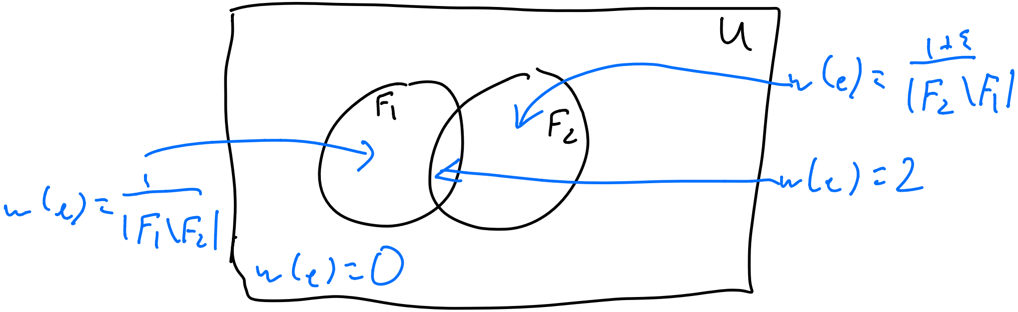

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Greedy not optimal: contradiction!