Lecture 14: Single-Source Shortest Paths

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601.433/633 Introduction to Algorithms

Introduction

Setup:

- Directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
- Length $\ell(\mathbf{x}, \mathbf{y})$ on each edge $(\mathbf{x}, \mathbf{y}) \in \mathbf{E}$ (equivalent: $\ell : \mathbf{E} \to \mathbb{R}$)
- Length of path $\mathbf{P}$ is $\ell(\mathbf{P}) = \sum_{(\mathbf{x},\mathbf{y}) \in \mathbf{P}} \ell(\mathbf{x}, \mathbf{y})$
- $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{x} \to \mathbf{y} \text{ paths } \mathbf{P}} \ell(\mathbf{P})$

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Today: source $\mathbf{v} \in \mathbf{V}$, want to compute shortest path from $\mathbf{v}$ to every $\mathbf{u} \in \mathbf{V}$

- $\mathbf{d}(\mathbf{u}) = \mathbf{d}(\mathbf{v}, \mathbf{u})$ for all $\mathbf{u} \in \mathbf{V}$
- Representation: "shortest path tree" out of $\mathbf{v}$.
- Often only care about distances – can reconstruct tree from distances.

Bellman-Ford

Dynamic Programming Approach

Subproblems:

- $\mathbf{OPT(u, i)}$: shortest path from $\mathbf{v}$ to $\mathbf{u}$ that uses at most $\mathbf{i}$ hops (edges)
- If no such path, set to "infinitely long" fake path.
- For simplicity, create loop (edge to and from the same node) at every node, length $\mathbf{0}$

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Theorem (Optimal Substructure)

$$\ell(\mathbf{OPT(u, k)}) = \begin{cases} \mathbf{0} & \textit{if } \mathbf{u = v, k = 0} \\ \infty & \textit{if } \mathbf{u \neq v, k = 0} \\ & \textit{otherwise} \end{cases}$$

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Proof of Optimal Substructure

$\mathbf{k = 0 :} \checkmark$. So let $\mathbf{k \geq 1}$.

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$\implies$ $\mathbf{OPT(x, k-1) \circ (x, u)}$ is a $\mathbf{v \to u}$ path with at most $\mathbf{k}$ edges, length $\ell(\mathbf{OPT(x, k-1)}) + \ell(\mathbf{x, u})$

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  $\implies$ $\mathbf{OPT(u, k)} \leq \mathbf{\min_{w:(w,u)\in E}}(\ell(\mathbf{OPT(w, k-1)}) + \ell(\mathbf{w, u}))$
- $\geq$: Let $\mathbf{z}$ be node before $\mathbf{u}$ in $\mathbf{OPT(u, k)}$, and let $\mathbf{P'}$ be the first $\mathbf{k-1}$ edges of $\mathbf{OPT(u, k)}$. Then

$$\begin{aligned}\mathbf{OPT(u,k)} &= \ell(\mathbf{P'}) + \ell(\mathbf{z,u}) \geq \ell(\mathbf{OPT(z,k-1)}) + \ell(\mathbf{z,u}) \\ &\geq \min_{\mathbf{w:(w,u)\in E}}(\ell(\mathbf{OPT(w,k-1)}) + \ell(\mathbf{w,u}))\end{aligned}$$

Bellman-Ford Algorithm

Obvious dynamic program!

$\mathbf{M}[\mathbf{u}, \mathbf{0}] = \infty$ for all $\mathbf{u} \in \mathbf{V}, \mathbf{u} \neq \mathbf{v}$
$\mathbf{M}[\mathbf{v}, \mathbf{0}] = \mathbf{0}$

for($\mathbf{k} = \mathbf{1}$ to $\mathbf{n} - \mathbf{1}$) {
    for($\mathbf{u} \in \mathbf{V}$) {
        $\mathbf{M}[\mathbf{u}, \mathbf{k}] = \mathbf{min}_{\mathbf{w}:(\mathbf{w},\mathbf{u}) \in \mathbf{E}} (\mathbf{M}[\mathbf{w}, \mathbf{k} - \mathbf{1}] + \ell(\mathbf{w}, \mathbf{u}))$
    }
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Running Time:

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Running Time:

- Obvious: $\mathbf{O(n^3)}$

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$\quad \}$

$\}$

Running Time:

- Obvious: $\mathbf{O(n^3)}$
- Smarter: $\mathbf{O(mn)}$

Bellman-Ford: Correctness

Theorem

After algorithm completes, $\mathbf{M[u,k]} = \ell(\mathbf{OPT(u,k)})$ *for all* $\mathbf{k \le n-1}$ *and* $\mathbf{u \in V}$.

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$$
\begin{aligned}
\mathbf{M[u,k]} &= \min_{\mathbf{w:(w,u)\in E}} (\mathbf{M[w,k-1]}) + \ell(\mathbf{w,u})) && \text{(algorithm)} \\
&= \min\nolimits_{\mathbf{w:(w,u)\in E}} (\ell(\mathbf{OPT(w,k-1)}) + \ell(\mathbf{w,u})) && \text{(induction)} \\
&= \ell(\mathbf{OPT(u,k)}) && \text{(optimal substructure)}
\end{aligned}
$$

□

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Detecting negative-weight cycle:

Negative Weights and Cycle

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- Negative-weight cycle: not really! Go around cycle forever, make distances arbitrarily negative
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Detecting negative-weight cycle: One more round of Bellman-Ford!

Relaxations

Common primitive in shortest path algorithms

- Reinterpret Bellman-Ford via relaxations
- Use relaxations for Dijkstra's algorithm

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$\hat{\mathbf{d}}(\mathbf{u})$: upper bound on $\mathbf{d}(\mathbf{u})$

- Initially: $\hat{\mathbf{d}}(\mathbf{v}) = \mathbf{0}$, $\hat{\mathbf{d}}(\mathbf{u}) = \infty$ for all $\mathbf{u} \neq \mathbf{v}$

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Intuition for relax($\mathbf{x}, \mathbf{y}$): can we improve $\hat{\mathbf{d}}(\mathbf{y})$ by going through $\mathbf{x}$?

```
relax(x, y) {
    if(d̂(y) > d̂(x) + ℓ(x, y)) {
        d̂(y) = d̂(x) + ℓ(x, y)
        y.parent = x
    }
}
```

Bellman-Ford as Relaxations

```
for(i = 1 to n) {
    foreach(u ∈ V) {
        foreach(edge (x, u)) {
            relax(x, u)
        }
    }
}
```

Bellman-Ford as Relaxations

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Not precisely the same: freezing/parallelism

Dijkstra's Algorithm

High Level

Intuition: "greedy starting at **v**"

- BFS but with edge lengths: use priority queue (heap) instead of queue!

Pros: faster than Bellman-Ford (super fast with appropriate data structures)

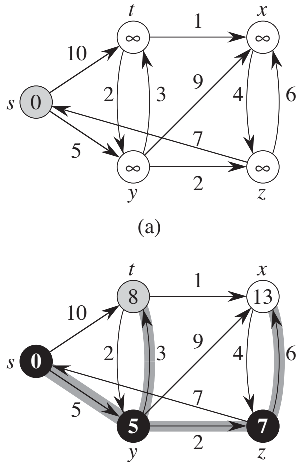
Cons: Doesn't work with negative edge weights.

Dijkstra's Algorithm

$\mathbf{T} = \emptyset$
$\hat{\mathbf{d}}(\mathbf{v}) = \mathbf{0}$
$\hat{\mathbf{d}}(\mathbf{u}) = \infty$ for all $\mathbf{u} \neq \mathbf{v}$

while(not all nodes in $\mathbf{T}$) {
    let $\mathbf{u}$ be node not in $\mathbf{T}$ with minimum $\hat{\mathbf{d}}(\mathbf{u})$
    Add $\mathbf{u}$ to $\mathbf{T}$
    foreach edge $(\mathbf{u}, \mathbf{x})$ with $\mathbf{x} \notin \mathbf{T}$ {
        relax(u,x)
    }
}

Dijkstra Example



Dijkstra Correctness

Theorem

Throughout the algorithm:

1. $\mathbf{T}$ *is a shortest-path tree from* $\mathbf{v}$ *to the nodes in* $\mathbf{T}$, *and*
2. $\hat{\mathbf{d}}(\mathbf{u}) = \mathbf{d}(\mathbf{u})$ *for every* $\mathbf{u} \in \mathbf{T}$.

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Proof. Induction on $|\mathbf{T}|$ (iterations of algorithm)

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Proof. Induction on $|\mathbf{T}|$ (iterations of algorithm)

Base Case: After first iteration (when $|\mathbf{T}| = \mathbf{1}$), added $\mathbf{v}$ to $\mathbf{T}$ with $\hat{\mathbf{d}}(\mathbf{v}) = \mathbf{d}(\mathbf{v}) = \mathbf{0}$ ✓

Correctness: Inductive Step (Sketch)

Consider iteration when $\mathbf{u}$ added to $\mathbf{T}$, let $\mathbf{w} = \mathbf{u.parent}$

$\implies \hat{\mathbf{d}}(\mathbf{u}) = \hat{\mathbf{d}}(\mathbf{w}) + \ell(\mathbf{w}, \mathbf{u}) = \mathbf{d}(\mathbf{w}) + \ell(\mathbf{w}, \mathbf{u})$ (induction)

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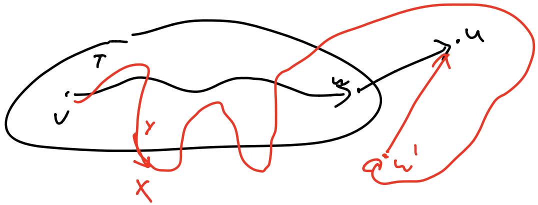

- Red path **P** actual shortest path, black path found by Dijkstra
- $\mathbf{w}'$ predecessor of **u** on **P**. Can't be in **T**.
  - If it was, would have $\hat{\mathbf{d}}(\mathbf{w}') = \mathbf{d}(\mathbf{w}')$ by induction, would have relaxed $(\mathbf{w}',\mathbf{u})$, so would have $\mathbf{w}' = \mathbf{u.parent}$
- **x** first node of **P** outside **T**, previous node **y**

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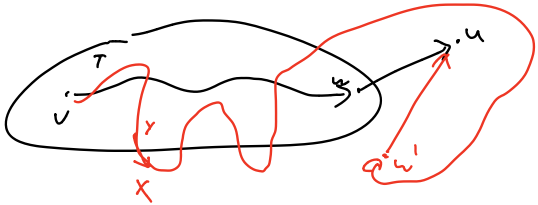

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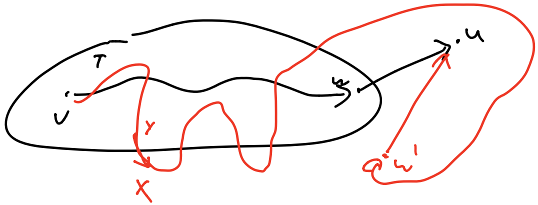

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Contradiction! Algorithm would have chosen $\mathbf{x}$ next, not $\mathbf{u}$.

Running Time

Algorithm needs to:

- Select node with minimum $\hat{\mathbf{d}}$ value $\mathbf{n}$ times
- Decrease a $\hat{\mathbf{d}}$ value at most once per relaxation $\implies \leq \mathbf{m}$ times.

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Nothing fancy, keep $\hat{\mathbf{d}}(\mathbf{u})$ in adjacency list: selecting min $\hat{\mathbf{d}}$ value takes $\mathbf{O}(\mathbf{n})$ time
$\implies \mathbf{O}(\mathbf{n}^2 + \mathbf{m}) = \mathbf{O}(\mathbf{n}^2)$ total.

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Keep $\hat{\mathbf{d}}$ values in a heap!

- Insert $\mathbf{n}$ times
- Extract-Min $\mathbf{n}$ times
- Decrease-Key $\mathbf{m}$ times

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Fibonacci Heap:

- Insert, Decrease-Key $\mathbf{O}(1)$ amortized
- Extract-Min $\mathbf{O}(\log \mathbf{n})$ amortized

$\implies \mathbf{O}(\mathbf{m} + \mathbf{n} \log \mathbf{n})$ running time