Lecture 12: Dynamic Programming II

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601.433/633 Introduction to Algorithms

Introduction

Today: two more examples of dynamic programming

- *Longest Common Subsequence* (strings)
- *Optimal Binary Search Tree* (trees)

Important problems, but really: more examples of dynamic programming

Both in CLRS (unlike Weighted Interval Scheduling)

Longest Common Subsequence

Definitions

String: Sequence of elements of some *alphabet* ($\{0,1\}$, or $\{A - Z\} \cup \{a - z\}$, etc.)

Definition: A sequence $Z = (z_1, \dots, z_k)$ is a *subsequence* of $X = (x_1, \dots, x_m)$ if there exists a strictly increasing sequence $(i_1, i_2, \dots, i_k)$ such that $x_{i_j} = z_j$ for all $j \in \{1, 2, \dots, k\}$.

Example: (B, C, D, B) is a subsequence of (A, B, C, B, D, A, B)

- Allowed to skip positions, unlike substring!

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Definition: In *Longest Common Subsequence* problem (LCS) we are given two strings $X = (x_1, \dots, x_m)$ and $Y = (y_1, \dots y_n)$. Need to find the longest Z which is a subsequence of both X and Y.

Subproblems

First and most important step of dynamic programming: define subproblems!

- Not obvious: **X** and **Y** might not even be same length!

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Prefixes of strings

- $\mathbf{X_i} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_i})$ (so $\mathbf{X} = \mathbf{X_m}$)
- $\mathbf{Y_j} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_j})$ (so $\mathbf{Y} = \mathbf{Y_n}$)

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Definition: Let $\mathbf{OPT(i, j)}$ be longest common subsequence of $\mathbf{X_i}$ and $\mathbf{Y_j}$

So looking for optimal solution $\mathbf{OPT = OPT(m, n)}$

- Last time $\mathbf{OPT}$ denotes value of solution, here denotes solution. Be flexible in notation

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Two-dimensional table!

Optimal Substructure

Second step of dynamic programming: prove optimal substructure

- Relationship between subproblems: show that solution to subproblem can be found from solutions to smaller subproblems

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1. *If* $\mathbf{x_i} = \mathbf{y_j}$*:*

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2. *If* $\mathbf{x}_i \neq \mathbf{y}_j$ *and* $\mathbf{z}_k \neq \mathbf{x}_i$*:*

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Optimal Substructure: Proof (I)

Case 1: If $\mathbf{x_i = y_j}$, then $\mathbf{z_k = x_i = y_j}$ and $\mathbf{Z_{k-1} = OPT(i-1, j-i)}$

Proof Sketch.

Contradiction.

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Part 2: Suppose $Z_{k-1} \neq OPT(i-1, j-1)$.

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Part 1: Suppose $\mathbf{x_i} = \mathbf{y_j} = \mathbf{a}$, but $\mathbf{z_k} \neq \mathbf{a}$. Add $\mathbf{a}$ to end of $\mathbf{Z}$, still have common subsequence, longer than LCS. Contradiction

Part 2: Suppose $\mathbf{Z_{k-1}} \neq \mathbf{OPT(i-1, j-1)}$.

$\Longrightarrow$ $\exists \mathbf{W}$ LCS of $\mathbf{X_{i-1}}, \mathbf{Y_{j-1}}$ of length $> \mathbf{k-1} \Longrightarrow \geq \mathbf{k}$

$\Longrightarrow$ $\mathbf{(W, a)}$ common subsequence of $\mathbf{X_i}, \mathbf{Y_j}$ of length $> \mathbf{k}$

- Contradiction to $\mathbf{Z}$ being LCS of $\mathbf{X_i}$ and $\mathbf{Y_j}$ $\square$

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Case 2: If $\mathbf{x_i} \neq \mathbf{y_j}$ and $\mathbf{z_k} \neq \mathbf{x_i}$ then $\mathbf{Z} = \mathbf{OPT(i-1, j)}$

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$\implies \mathbf{Z} = \mathbf{OPT(i-1, j)}$ □

Optimal Substructure: Proof (III)

Case 3: If $\mathbf{x_i \neq y_j}$ and $\mathbf{z_k \neq y_j}$ then $\mathbf{Z = OPT(i, j-1)}$

Proof.

Symmetric to Case 2. □

Structure Corollary

Corollary

$$\mathrm{OPT}(i,j) = \begin{cases} \varnothing & \textit{if } i = 0 \textit{ or } j = 0, \\ \mathrm{OPT}(i-1, j-1) \circ x_i & \textit{if } i,j > 0 \textit{ and } x_i = y_j \\ \max(\mathrm{OPT}(i, j-1), \mathrm{OPT}(i-1, j)) & \textit{if } i,j > 0 \textit{ and } x_i \neq y_j \end{cases}$$

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Gives obvious recursive algorithm

- Can take exponential time (good exercise at home!)

Dynamic Programming!

- Top-Down: are problems getting "smaller"? What does "smaller" mean?
- Bottom-Up: two-dimensional table! What order to fill it in?

Dynamic Programming Algorithm

```
LCS(X,Y) {
    for($\mathbf{i = 0}$ to $\mathbf{m}$) $\mathbf{M[i,0] = 0}$;
    for($\mathbf{j = 0}$ to $\mathbf{n}$) $\mathbf{M[0,j] = 0}$;
    for($\mathbf{i = 1}$ to $\mathbf{m}$) {
        for($\mathbf{j = 1}$ to $\mathbf{n}$) {
            if($\mathbf{x_i = y_j}$)
                $\mathbf{M[i,j] = 1 + M[i-1,j-1]}$;
            else
                $\mathbf{M[i,j] = max(M[i,j-1], M[i-1,j])}$;
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        }
    return $\mathbf{M[m,n]}$;
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Running Time: O(mn)

Correctness

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Induction on $\mathbf{i + j}$ (or could do on iterations in the algorithm)

Base Case: $\mathbf{i + j = 0 \implies i = j = 0 \implies M[i,j] = 0 = |OPT(i,j)|}$

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Proof.

Induction on $\mathbf{i+j}$ (or could do on iterations in the algorithm)

Base Case: $\mathbf{i+j=0 \implies i=j=0 \implies M[i,j]=0=|OPT(i,j)|}$

Inductive Step: Divide into three cases

1. If $\mathbf{i=0}$ or $\mathbf{j=0}$, then $\mathbf{M[i,j]=0=|OPT(i,j)|}$

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3. If $\mathbf{x_i \neq y_j}$, then

$$\begin{aligned}
\mathbf{M[i,j]} &= \mathbf{\max(M[i,j-1],M[i-1,j])} && \text{(def of algorithm)}\\
&= \mathbf{\max(|OPT(i,j-1)|,|OPT(i-1,j)|)} && \text{(induction)}\\
&= \mathbf{|OPT(i,j)|} && \text{(structure thm/corollary)}
\end{aligned}$$

Computing a Solution

Like we talked about last lecture: backtrack through dynamic programming table.

Details in CLRS 15.4

Optimal Binary Search Trees

Problem Definition

Input: probability distribution / search frequency of keys

- $\mathbf{n}$ distinct keys $\mathbf{k_1 < k_2 < \cdots < k_n}$
- For each $\mathbf{i} \in [\mathbf{n}]$, probability $\mathbf{p_i}$ that we search for $\mathbf{k_i}$ (so $\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{p_i} = \mathbf{1}$)

What's the best binary search tree for these keys and frequencies?

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Cost of searching for k_i in tree T is $\text{depth}_T(k_i) + 1$ (say depth of root = 0)

$\implies E[\text{cost of search in } T] = \sum_{i=1}^{n} p_i(\text{depth}_T(k_i) + 1)$

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Cost of searching for $\mathbf{k_i}$ in tree $\mathbf{T}$ is $\mathbf{depth_T(k_i) + 1}$ (say depth of root = $\mathbf{0}$)
$\Longrightarrow$ $\mathbf{E}[\text{cost of search in } \mathbf{T}] = \sum_{i=1}^{n} \mathbf{p_i(depth_T(k_i) + 1)}$

Definition: $\mathbf{c(T) = \sum_{i=1}^{n} p_i(depth_T(k_i) + 1)}$

Problem: Find search tree $\mathbf{T}$ minimizing cost.

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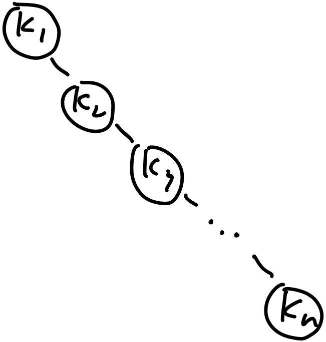

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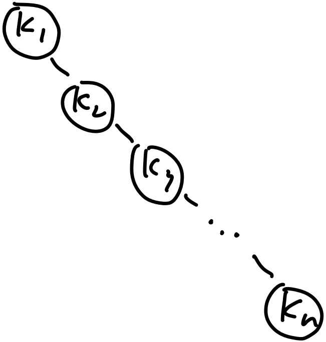

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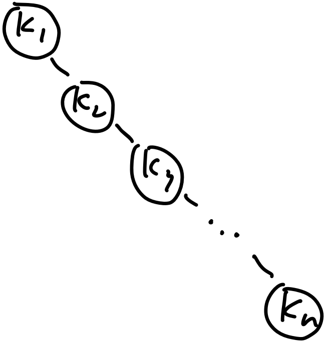

$\mathbf{E}[\text{cost of search in } \mathbf{T}] \geq \mathbf{\Omega(n)}$

Balanced search tree: $\mathbf{E}[\text{cost}] \leq \mathbf{O(\log n)}$

Intuition

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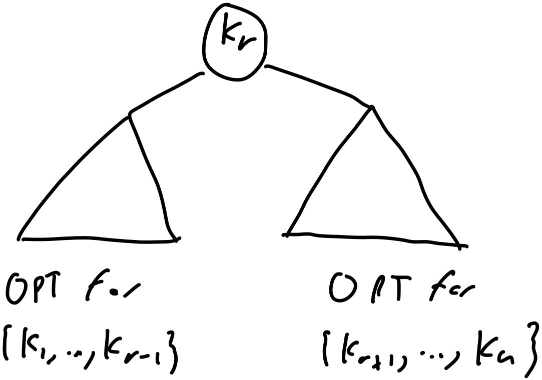

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Let $\mathbf{OPT(i,j)}$ with $\mathbf{i \le j}$ be optimal tree for keys $\{\mathbf{k_i, k_{i+1}, \ldots, k_j}\}$: tree $\mathbf{T}$ minimizing $\mathbf{c(T) = \sum_{a=i}^{j} p_a(depth_T(k_a) + 1)}$

By convention, if $\mathbf{i > j}$ then $\mathbf{OPT(i,j)}$ empty
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Theorem (Optimal Substructure)

Let $\mathbf{k_r}$ *be the root of* $\mathbf{OPT(i,j)}$*. Then the left subtree of* $\mathbf{OPT(i,j)}$ *is* $\mathbf{OPT(i,r-1)}$*, and the right subtree of* $\mathbf{OPT(i,j)}$ *is* $\mathbf{OPT(r+1,j)}$*.*

Proof Sketch of Optimal Substructure

Definitions:

- Let $\mathbf{T} = \mathbf{OPT(i, j)}$, $\mathbf{T_L}$ its left subtree, $\mathbf{T_R}$ its right subtree.
- Suppose for contradiction $\mathbf{T_L} \neq \mathbf{OPT(i, r-1)}$, let $\mathbf{T'} = \mathbf{OPT(i, r-1)}$

  $\implies$ $\mathbf{c(T') < c(T_L)}$ (def of $\mathbf{OPT(i, r-1)}$)
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Contradicts $\mathbf{T} = \mathbf{OPT(i, j)}$

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Contradicts $\mathbf{T = OPT(i,j)}$

Symmetric argument works for $\mathbf{T_R = OPT(r+1, j)}$

Cost Corollary

Corollary

$c(OPT(i,j)) = \sum_{a=i}^{j} p_a + \min_{i \le r \le j}(c(OPT(i,r-1)) + c(OPT(r+1,j)))$

Let k_r be root of $OPT(i,j)$

$$\begin{aligned} c(OPT(i,j)) &= \sum_{a=i}^{j} p_a(\text{depth}_{OPT(i,j)}(k_a)+1) \\ &= \sum_{a=i}^{r-1}(p_a(\text{depth}_{OPT(i,r-1)}(k_a)+2)) + p_r + \sum_{a=r+1}^{j} p_a(\text{depth}_{OPT(r+1,j)}(k_a)+2) \\ &= \sum_{a=i}^{j} p_a + \sum_{a=i}^{r-1}(p_a(\text{depth}_{OPT(i,r-1)}(k_a)+1)) + \sum_{a=r+1}^{j} p_a(\text{depth}_{OPT(r+1,j)}(k_a)+1) \\ &= \sum_{a=i}^{j} p_a + c(OPT(i,r-1)) + c(OPT(r+1,j)). \end{aligned}$$

Cost Corollary

Corollary

$c(OPT(i,j)) = \sum_{a=i}^{j} p_a + \min_{i \le r \le j}(c(OPT(i,r-1)) + c(OPT(r+1,j)))$

Let k_r be root of $OPT(i,j)$

$$\begin{aligned} c(OPT(i,j)) &= \sum_{a=i}^{j} p_a(\text{depth}_{OPT(i,j)}(k_a)+1) \\ &= \sum_{a=i}^{r-1}(p_a(\text{depth}_{OPT(i,r-1)}(k_a)+2)) + p_r + \sum_{a=r+1}^{j} p_a(\text{depth}_{OPT(r+1,j)}(k_a)+2) \\ &= \sum_{a=i}^{j} p_a + \sum_{a=i}^{r-1}(p_a(\text{depth}_{OPT(i,r-1)}(k_a)+1)) + \sum_{a=r+1}^{j} p_a(\text{depth}_{OPT(r+1,j)}(k_a)+1) \\ &= \sum_{a=i}^{j} p_a + c(OPT(i,r-1)) + c(OPT(r+1,j)). \end{aligned}$$

Same logic holds for any possible root $\implies$ take min

Algorithm

Fill in table **M**:

$$\mathbf{M}[\mathbf{i},\mathbf{j}] = \begin{cases} \mathbf{0} & \text{if } \mathbf{i} > \mathbf{j} \\ \mathbf{min}_{\mathbf{i} \le \mathbf{r} \le \mathbf{j}} \left(\sum_{\mathbf{a}=\mathbf{i}}^{\mathbf{j}} \mathbf{p}_{\mathbf{a}} + \mathbf{M}[\mathbf{i}, \mathbf{r}-\mathbf{1}] + \mathbf{M}[\mathbf{r}+\mathbf{1}, \mathbf{j}] \right) & \text{if } \mathbf{i} \le \mathbf{j} \end{cases}$$

Algorithm

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$$\mathbf{M}[\mathbf{i},\mathbf{j}] = \begin{cases} \mathbf{0} & \text{if } \mathbf{i} > \mathbf{j} \\ \mathbf{min}_{\mathbf{i} \le \mathbf{r} \le \mathbf{j}} \left(\sum_{\mathbf{a}=\mathbf{i}}^{\mathbf{j}} \mathbf{p}_{\mathbf{a}} + \mathbf{M}[\mathbf{i},\mathbf{r}-\mathbf{1}] + \mathbf{M}[\mathbf{r}+\mathbf{1},\mathbf{j}] \right) & \text{if } \mathbf{i} \le \mathbf{j} \end{cases}$$

Top-Down (memoization): are problems getting smaller?

Algorithm

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Top-Down (memoization): are problems getting smaller? Yes! $\mathbf{j} - \mathbf{i}$ decreases in every recursive call.

Algorithm

Fill in table $\mathbf{M}$:

$$\mathbf{M}[\mathbf{i},\mathbf{j}] = \begin{cases} \mathbf{0} & \text{if } \mathbf{i} > \mathbf{j} \\ \min_{\mathbf{i} \le \mathbf{r} \le \mathbf{j}} \left(\sum_{\mathbf{a}=\mathbf{i}}^{\mathbf{j}} \mathbf{p_a} + \mathbf{M}[\mathbf{i}, \mathbf{r}-\mathbf{1}] + \mathbf{M}[\mathbf{r}+\mathbf{1}, \mathbf{j}] \right) & \text{if } \mathbf{i} \le \mathbf{j} \end{cases}$$

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Correctness. Claim $\mathbf{M}[\mathbf{i},\mathbf{j}] = \mathbf{c}(\mathbf{OPT}(\mathbf{i},\mathbf{j}))$. Induction on $\mathbf{j} - \mathbf{i}$.

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Correctness. Claim $\mathbf{M}[\mathbf{i},\mathbf{j}] = \mathbf{c}(\mathbf{OPT}(\mathbf{i},\mathbf{j}))$. Induction on $\mathbf{j} - \mathbf{i}$.

- Base case: if $\mathbf{j} - \mathbf{i} < \mathbf{0}$ then $\mathbf{M}[\mathbf{i},\mathbf{j}] = \mathbf{OPT}(\mathbf{i},\mathbf{j}) = \mathbf{0}$

Algorithm

Fill in table **M**:

$$\mathbf{M}[\mathbf{i},\mathbf{j}] = \begin{cases} \mathbf{0} & \text{if } \mathbf{i} > \mathbf{j} \\ \min_{\mathbf{i}\le \mathbf{r}\le \mathbf{j}} \left(\sum_{\mathbf{a}=\mathbf{i}}^{\mathbf{j}} \mathbf{p}_\mathbf{a} + \mathbf{M}[\mathbf{i},\mathbf{r}-\mathbf{1}] + \mathbf{M}[\mathbf{r}+\mathbf{1},\mathbf{j}] \right) & \text{if } \mathbf{i} \le \mathbf{j} \end{cases}$$

Top-Down (memoization): are problems getting smaller? Yes! $\mathbf{j}-\mathbf{i}$ decreases in every recursive call.

Correctness. Claim $\mathbf{M}[\mathbf{i},\mathbf{j}] = \mathbf{c}(\mathbf{OPT}(\mathbf{i},\mathbf{j}))$. Induction on $\mathbf{j}-\mathbf{i}$.

- Base case: if $\mathbf{j}-\mathbf{i}<\mathbf{0}$ then $\mathbf{M}[\mathbf{i},\mathbf{j}] = \mathbf{OPT}(\mathbf{i},\mathbf{j}) = \mathbf{0}$
- Inductive step:

$$\begin{aligned} \mathbf{M}[\mathbf{i},\mathbf{j}] &= \min_{\mathbf{i}\le \mathbf{r}\le \mathbf{j}} \left(\sum_{\mathbf{a}=\mathbf{i}}^{\mathbf{j}} \mathbf{p}_\mathbf{a} + \mathbf{M}[\mathbf{i},\mathbf{r}-\mathbf{1}] + \mathbf{M}[\mathbf{r}+\mathbf{1},\mathbf{j}] \right) && \text{(alg def)} \\ &= \min_{\mathbf{i}\le \mathbf{r}\le \mathbf{j}} \left(\sum_{\mathbf{a}=\mathbf{i}}^{\mathbf{j}} \mathbf{p}_\mathbf{a} + \mathbf{c}(\mathbf{OPT}(\mathbf{i},\mathbf{r}-\mathbf{1})) + \mathbf{c}(\mathbf{OPT}(\mathbf{r}+\mathbf{1},\mathbf{j})) \right) && \text{(induction)} \\ &= \mathbf{c}(\mathbf{OPT}(\mathbf{i},\mathbf{j})) && \text{(cost corollary)} \end{aligned}$$

Algorithm: Bottom-up

What order to fill the table in?

- Obvious approach: for($\mathbf{i} = \mathbf{1}$ to $\mathbf{n} - \mathbf{1}$) for($\mathbf{j} = \mathbf{i} + \mathbf{1}$ to $\mathbf{n}$) Doesn't work!

Algorithm: Bottom-up

What order to fill the table in?

- Obvious approach: for($i = 1$ to $n - 1$) for($j = i + 1$ to n) Doesn't work!
- Take hint from induction: $j - i$

```
OBST {
    Set $M[i, j] = 0$ for all $j > i$;
    Set $M[i, i] = p_i$ for all $i$
    for($\ell = 1$ to $n - 1$) {
        for($i = 1$ to $n - \ell$) {
            $j = i + \ell$
            $M[i, j] = \min_{i \le r \le j} \left( \sum_{a=i}^{j} p_a + M[i, r - 1] + M[r + 1, j] \right)$;
        }
    }
    return $M[1, n]$;
}
```

Analysis

Correctness: same as top-down

Running Time:

Analysis

Correctness: same as top-down

Running Time:

- # table entries:

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $\mathbf{O(n^2)}$

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $\mathbf{O(n^2)}$
- Time to compute table entry $\mathbf{M[i,j]}$:

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $\mathbf{O(n^2)}$
- Time to compute table entry $\mathbf{M[i,j]}$: $\mathbf{O(j-i) = O(n)}$

Analysis

Correctness: same as top-down

Running Time:

- # table entries: $\mathbf{O(n^2)}$
- Time to compute table entry $\mathbf{M[i,j]}$: $\mathbf{O(j-i) = O(n)}$

Total running time: $\mathbf{O(n^3)}$