Lasserre Integrality Gaps for Graph Spanners and Related Problems

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Graph Spanners

This talk is about *spanners*: graph sparsifiers that (approximately) preserve distances.

Given graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, subgraph $\mathbf{H}$ of $\mathbf{G}$ is a $\mathbf{t}$-*spanner* of $\mathbf{G}$ if

$$\mathbf{d_H}(\mathbf{u}, \mathbf{v}) \leq \mathbf{t} \cdot \mathbf{d_G}(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

- $\mathbf{t}$ is the *stretch* of the spanner.
- In this paper: $\mathbf{G}$ unweighted, connected
- Sufficient for stretch condition to hold for all edges $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$

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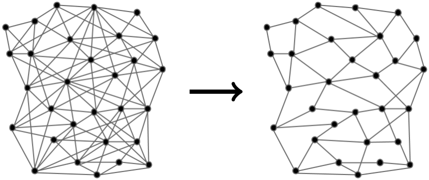

Original graph $\mathbf{G}$ A $\mathbf{3}$-spanner $\mathbf{H}$

Existential Bounds

Want to have small stretch, few edges.

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Fundamental Theorem of Graph Spanners [Althöfer et al '93]:

- For any positive integer $\mathbf{k}$, all graphs have a $\mathbf{(2k-1)}$-spanner with $\mathbf{O(n^{1+1/k})}$ edges (which can be found in polynomial time via a greedy algorithm), and
- There exist graphs in which all $\mathbf{(2k-1)}$-spanners have $\mathbf{\Omega(n^{1+1/k})}$ edges (assuming **Erdös Girth Conjecture**).

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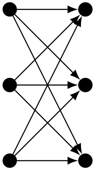
No such theorem possible for directed graphs!

$\mathbf{K_{n,n}}$: Removing any edge causes infinite stretch



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Given $\mathbf{G}, \mathbf{t}$, efficient algorithm for finding *best* $\mathbf{t}$-spanner of $\mathbf{G}$?

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Approximation algorithms instead of extremal graph theory!

- Difficulty caused by requiring *efficient* algorithms: if don't care about computation, just try all subgraphs!

State of the Art

Two variants: undirected and directed

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BASIC **t**-SPANNER: **G** undirected

- "Generic" approximation:
  - $\mathbf{G}$ connected, so $\mathbf{OPT} \geq \mathbf{n} - \mathbf{1}$
  - When $\mathbf{t} \in \{\mathbf{2k} - \mathbf{1}, \mathbf{2k}\}$, Fundamental Theorem $\Longrightarrow$ $\mathbf{ALG} \leq \mathbf{O}(\mathbf{n}^{\mathbf{1+1/k}})$
  - So greedy is an $\mathbf{O}(\mathbf{n}^{\mathbf{1/k}})$ approximation.

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- Non-generic approximations:
  - $\mathbf{t} = \mathbf{3}$: $\mathbf{O}(\mathbf{n}^{1/3})$ [BBMRY '13]
  - $\mathbf{t} = \mathbf{4}$: $\mathbf{O}(\mathbf{n}^{1/3})$ [D-Zhang '16]

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  - Inapproximable to within $\mathbf{2^{\frac{\log^{1-\epsilon} n}{t}}}$ [D-Kortsarz-Raz '12]

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DIRECTED $\mathbf{t}$-SPANNER

- No generic approximation!
- Non-generic:
  - $\mathbf{t = 3, 4}$: same as undirected
  - $\mathbf{\tilde{O}(n^{1-1/t})}$ [BGJRW '09]
  - $\mathbf{\tilde{O}(n^{2/3})}$ [D-Krauthgamer '11]
  - $\mathbf{\tilde{O}(\sqrt{n})}$ [BBMRY '13]
- Hardness:
  - Inapproximable to within $\mathbf{2^{\log^{1-\epsilon} n}}$ [Kortsarz '01, Elkin-Peleg '07]

Integrality Gaps

Many open questions:

- For undirected, can we do better than generic for $\mathbf{t \geq 5}$? How much better?
- For directed: better than $\tilde{\mathbf{O}}(\sqrt{\mathbf{n}})$ when $\mathbf{t \geq 5}$? Better than $\tilde{\mathbf{O}}(\mathbf{n^{1/3}})$?

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Theorem [D-Krauthgamer '11]: The integrality gap of the basic LP for Directed $\mathbf{t}$-Spanner is at least $\mathbf{\Omega(n^{1/3-\epsilon})}$.

Theorem [D-Zhang '16]: The integrality gap of the basic LP for Basic $\mathbf{(2k-1)}$-Spanner is at least $\mathbf{\Omega(n^{\frac{1}{(1+\epsilon)(k+4)}})}$.

Better Relaxations?

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LP/SDP hierarchies (lift-and-project)

- Given base LP, define a "level-$\mathbf{r}$" LP/SDP which strengthens original LP.
- Takes time $\mathbf{n}^{\mathbf{O}(\mathbf{r})}$ to solve
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Famous hierarchies:

- Lovasz-Schrijver '91 (LP hierarchy)
- Sherali-Adams '90 (LP hierarchy)
- Lasserre '01 / "Sum-of-Squares" (SDP hierarchy)

Results

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Theorem: For every constant $\mathbf{0} < \varepsilon < \mathbf{1}$, the integrality gap of the $\mathbf{n}^{\mathbf{\Omega}(\varepsilon)}$-th level Lasserre SDP for Directed $(\mathbf{2k}-\mathbf{1})$-Spanner is at least $\left(\frac{\mathbf{n}}{\mathbf{k}}\right)^{\frac{1}{18}-\mathbf{\Theta}(\varepsilon)}$.

Theorem: For every constant $\mathbf{0} < \varepsilon < \mathbf{1}$, the integrality gap of the $\mathbf{n}^{\mathbf{\Omega}(\varepsilon)}$-th level Lasserre SDP for Basic $(\mathbf{2k}-\mathbf{1})$-Spanner is at least $\frac{\mathbf{1}}{\mathbf{k}} \cdot \left(\frac{\mathbf{n}}{\mathbf{k}}\right)^{\min\left\{\frac{1}{18}, \frac{5}{32k-6}\right\}-\mathbf{\Theta}(\varepsilon)} = \mathbf{n}^{\mathbf{\Theta}\left(\frac{1}{k}-\varepsilon\right)}$.

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Can extend to similar results for Directed Steiner Network and Shallow-Light Steiner Network.

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Most easily written as a "length-bounded flow" LP (originally introduced by [D-Krauthgamer '11])

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Equivalent LP with polynomial variables:

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- Let $\mathcal{Z}^{\mathbf{u},\mathbf{v}} = \{\mathbf{z} \in [0,1]^{|\mathbf{E}|} : \sum_{\mathbf{e} \in \mathbf{P}} \mathbf{z}_{\mathbf{e}} \geq 1 \ \forall \mathbf{P} \in \mathcal{P}_{\mathbf{u},\mathbf{v}}\}$
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Infinite constraints, only finite of them matter.

Lasserre SDP

Variables: $\mathbf{y_S}$ for all $\mathbf{S} \subseteq \mathbf{E}$ with $|\mathbf{S}| \leq \mathbf{r}$ (where $\mathbf{r}$ is the level)

Intuition:

- Add constraints which force $\mathbf{y_S}$ to "act like" $\prod_{\mathbf{e} \in \mathbf{S}} \mathbf{y_e}$
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$$\begin{aligned}
\min \quad & \sum_{e \in E} y_e \\
\text{s.t.} \quad & y_{\emptyset} = 1 \\
& M_{r+1}(y) := (y_{I \cup J})_{|I|,|J| \leq r+1} \succcurlyeq 0 \\
& M_r^z(y) := \left(\sum_{e \in E} z_e y_{I \cup J \cup \{e\}} - y_{I \cup J} \right)_{|I|,|J| \leq r} \succcurlyeq 0 \quad \forall (u,v) \in E, z \in \mathcal{Z}^{u,v}
\end{aligned}$$

Old Gaps

Hardness results [Kortsarz '01, EP '07, DKR '12]: reduce from LABEL COVER / MIN-REP

- If LABEL COVER instance has a good solution then so does our spanner instance
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- *Every* instance of UNIQUE GAMES has good fractional solution after reduction
  - Not true of Lasserre SDP! Lasserre works well for UNIQUE GAMES.

High-Level Gap Idea(s)

Instead of Unique Games, use Projection Games (different special case of Label Cover)

- Integrality gaps against Lasserre for Projection Games given recently by [Chlamtáč, Manurangsi, Moshkovitz, Vijayaraghavan '17]!
- Apply hardness reduction to spanners to these instances.

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- No good integral solution: requires using an extra property of instance of [CMMV '17] (not too hard)
- Cheap fractional solution: surprisingly tricky!
  - Start with fractional solution to Projection Games instance, extend to spanner instance
  - Prove matrices are PSD through careful decompositions, analyze each part separately

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Open questions:

- Better approximations for any of these problems!
- Better base LP? Some other technique?

Thanks!