Partially Optimal Edge Fault-Tolerant Spanners

Greg Bodwin[1] Michael Dinitz[2] Caleb Robelle[3]

[1]University of Michigan

[2]Johns Hopkins University

[3]MIT

SODA '22

Graph Spanners: Basics

Definition

Given graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, subgraph $\mathbf{H}$ of $\mathbf{G}$ is a $\mathbf{t}$-*spanner* of $\mathbf{G}$ if

$$\mathbf{d_H(u, v) \leq t \cdot d_G(u, v)} \quad \text{for all } \mathbf{u, v} \in \mathbf{V}$$

- $\mathbf{t}$ is the *stretch* of the spanner.
- In this paper: $\mathbf{G}$ undirected, connected
- Sufficient for stretch condition to hold for all edges $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$

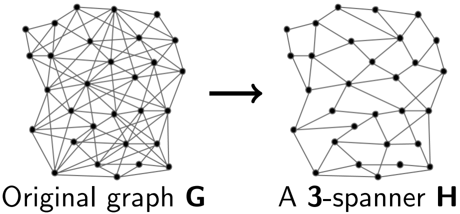
Graph Spanners: Basics

Definition

Given graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, subgraph $\mathbf{H}$ of $\mathbf{G}$ is a $\mathbf{t}$-*spanner* of $\mathbf{G}$ if

$$\mathbf{d_H}(\mathbf{u}, \mathbf{v}) \leq \mathbf{t} \cdot \mathbf{d_G}(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

- $\mathbf{t}$ is the *stretch* of the spanner.
- In this paper: $\mathbf{G}$ undirected, connected
- Sufficient for stretch condition to hold for all edges $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$



Main Theorem

Theorem (Althöfer et al '93)

- *For any positive integer* $\mathbf{k}$*, all graphs have a* $\mathbf{(2k-1)}$*-spanner with* $\mathbf{O(n^{1+1/k})}$ *edges, and*
- *There exist graphs in which all* $\mathbf{(2k-1)}$*-spanners have* $\mathbf{\Omega(n^{1+1/k})}$ *edges (assuming* **Erdős Girth Conjecture***).*

Main Theorem

Theorem (Althöfer et al '93)

- *For any positive integer* $\mathbf{k}$, *all graphs have a* $\mathbf{(2k-1)}$*-spanner with* $\mathbf{O(n^{1+1/k})}$ *edges, and*
- *There exist graphs in which all* $\mathbf{(2k-1)}$*-spanners have* $\mathbf{\Omega(n^{1+1/k})}$ *edges (assuming* **Erdős Girth Conjecture***).*

Upper bound statement existential, but actually algorithmic: *greedy algorithm*

$\mathbf{H \leftarrow (V, \varnothing)}$
for all $\mathbf{\{u,v\} \in E}$ in *nondecreasing* weight order **do**
  if $\mathbf{d_H(u,v) > (2k-1) \cdot w(u,v)}$ **then**
    add $\{\mathbf{u},\mathbf{v}\}$ to H
  end if
end for
return H

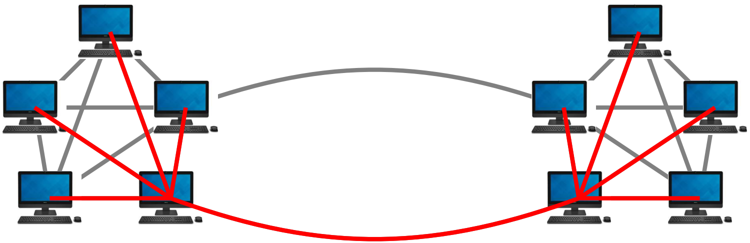
Spanners For Distributed Systems

Go back to # edges.

In Theory, we're done. We have a simple, optimal, textbook algorithm.

In Practice, spanners useful in many applications, but commonly used in distributed systems.

Imagine we build a **3-spanner** of this network of computers, which need to talk to each other



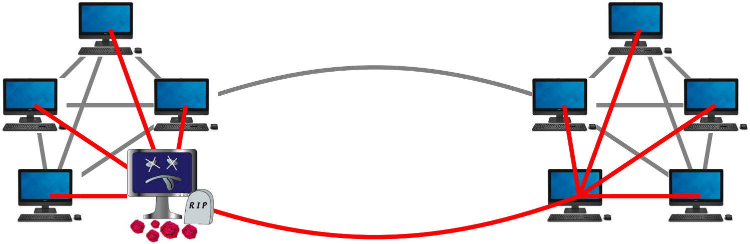
Spanners For Distributed Systems

Go back to # edges.

In Theory, we're done. We have a simple, optimal, textbook algorithm.

In Practice, spanners useful in many applications, but commonly used in distributed systems.

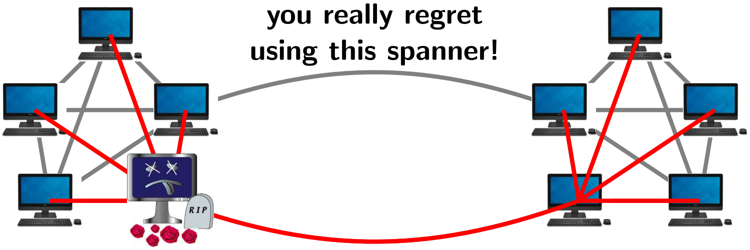
Imagine we build a **3-spanner** of this network of computers, which need to talk to each other . . . **but then one breaks.**



Spanners For Distributed Systems

Go back to # edges.
In Theory, we're done. We have a simple, optimal, textbook algorithm.
In Practice, spanners useful in many applications, but commonly used in distributed systems.

Imagine we build a **3-spanner** of this network of computers, which need to talk to each other
. . . **but then one breaks.**



Fault-Tolerant Spanners

Definition (Chechik, Langberg, Peleg, Roditty '09)

A subgraph $\mathbf{H} \subseteq \mathbf{G}$ is an $\mathbf{f}$-*Edge Fault Tolerant (EFT)* $\mathbf{(2k-1)}$-*spanner* of $\mathbf{G}$ if, for every possible set $\mathbf{F}$ of $|\mathbf{F}| = \mathbf{f}$ edges, we have

$$\mathbf{H} \setminus \mathbf{F} \text{ is a } (\mathbf{2k-1})\text{-spanner of } \mathbf{G} \setminus \mathbf{F}.$$

Equivalently: for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $\mathbf{F} \subseteq \mathbf{E}$ with $|\mathbf{F}| \leq \mathbf{f}$,

$$\mathbf{d}_{\mathbf{H} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v}) \leq (\mathbf{2k-1}) \cdot \mathbf{d}_{\mathbf{G} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v})$$

$\mathbf{f}$-*Vertex Fault Tolerant* ($\mathbf{f}$-*VFT*): $\mathbf{F} \subseteq \mathbf{V}$.

Fault-Tolerant Spanners

Definition (Chechik, Langberg, Peleg, Roditty '09)

A subgraph $\mathbf{H} \subseteq \mathbf{G}$ is an $\mathbf{f}$*-Edge Fault Tolerant (EFT)* $\mathbf{(2k-1)}$*-spanner* of $\mathbf{G}$ if, for every possible set $\mathbf{F}$ of $|\mathbf{F}| = \mathbf{f}$ edges, we have

$$\mathbf{H} \setminus \mathbf{F} \text{ is a } \mathbf{(2k-1)}\text{-spanner of } \mathbf{G} \setminus \mathbf{F}.$$

Equivalently: for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $\mathbf{F} \subseteq \mathbf{E}$ with $|\mathbf{F}| \leq \mathbf{f}$,

$$\mathbf{d}_{\mathbf{H} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v}) \leq (\mathbf{2k-1}) \cdot \mathbf{d}_{\mathbf{G} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v})$$

$\mathbf{f}$*-Vertex Fault Tolerant (*$\mathbf{f}$*-VFT):* $\mathbf{F} \subseteq \mathbf{V}$.

Subtle definition: $\mathbf{H}$ only has to be "fault-tolerant" if $\mathbf{G}$ is "fault-tolerant"

- *Relative* fault-tolerance

Fault-Tolerant Spanners

Question: How much "extra" above $\mathbf{n^{1+1/k}}$ do we need to pay for $\mathbf{f}$-fault tolerance?

Fault-Tolerant Spanners

Question: How much "extra" above $\mathbf{n^{1+1/k}}$ do we need to pay for $\mathbf{f}$-fault tolerance?

Reasonable intuition:

- Natural approach: redundancy. Build a bunch of different spanners so that for all $\mathbf{F}$, at least one spanner is unaffected
- Needs at least $\mathbf{f+1}$ redundancy, pay extra factor of $\mathbf{f}$

Fault-Tolerant Spanners

Question: How much "extra" above $\mathbf{n^{1+1/k}}$ do we need to pay for $\mathbf{f}$-fault tolerance?

Reasonable intuition:

- Natural approach: redundancy. Build a bunch of different spanners so that for all $\mathbf{F}$, at least one spanner is unaffected
- Needs at least $\mathbf{f+1}$ redundancy, pay extra factor of $\mathbf{f}$

Theorem (Bodwin, D, Parter, Vassilevska Williams '18)

Existential lower bounds on $\mathbf{f}$*-fault tolerance:*

- $\mathbf{f}$*-VFT* $\mathbf{(2k-1)}$*-spanner:* $\mathbf{\Omega\left(f^{1-1/k} n^{1+1/k}\right)}$ *edges.*
- $\mathbf{f}$*-EFT* $\mathbf{(2k-1)}$*-spanner:*
  - $\mathbf{k=2}$*:* $\mathbf{\Omega\left(f^{1-1/k} n^{1+1/k}\right) = \Omega\left(f^{1/2} n^{3/2}\right)}$ *edges.*
  - $\mathbf{k \geq 3}$*:* $\mathbf{\Omega\left(f^{\frac{1}{2}(1-1/k)} n^{1+1/k} + fn\right)}$ *edges.*

Vertex Fault-Tolerant Spanner Bounds

Spanner Size	Runtime	Greedy?	Citation
$\tilde{O}\left(k^{O(f)} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(k^{O(f)} \cdot n^{3+1/k}\right)$		[CLPR '10]
$\tilde{O}\left(f^{2-1/k} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(f^{2-2/k} \cdot mn^{1+1/k}\right)$		[DK '11]
$O\left(\exp(k) f^{1-1/k} \cdot n^{1+1/k}\right)$	$O\left(\exp(k) \cdot mn^{O(f)}\right)$	✓	[BDPV '18]
$O\left(f^{1-1/k} \cdot n^{1+1/k}\right)$	$O\left(mn^{O(f)}\right)$	✓	[BP '19]
$O\left(kf^{1-1/k} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(f^{2-1/k} \cdot mn^{1+1/k}\right)$	✓*	[DR '20]
$O\left(f^{1-1/k} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(f^{1-1/k} n^{2+1/k} + mf^2\right)$	✓*	[BDR '21]

Vertex Fault-Tolerant Spanner Bounds

Spanner Size	Runtime	Greedy?	Citation
$\tilde{O}\left(k^{O(f)} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(k^{O(f)} \cdot n^{3+1/k}\right)$		[CLPR '10]
$\tilde{O}\left(f^{2-1/k} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(f^{2-2/k} \cdot mn^{1+1/k}\right)$		[DK '11]
$O\left(\exp(k) f^{1-1/k} \cdot n^{1+1/k}\right)$	$O\left(\exp(k) \cdot mn^{O(f)}\right)$	✓	[BDPV '18]
$O\left(f^{1-1/k} \cdot n^{1+1/k}\right)$	$O\left(mn^{O(f)}\right)$	✓	[BP '19]
$O\left(kf^{1-1/k} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(f^{2-1/k} \cdot mn^{1+1/k}\right)$	✓*	[DR '20]
$O\left(f^{1-1/k} \cdot n^{1+1/k}\right)$	$\tilde{O}\left(f^{1-1/k} n^{2+1/k} + mf^2\right)$	✓*	[BDR '21]

So VFT essentially resolved!

Edge Fault-Tolerant Spanner Bounds

What about edge fault-tolerance?

- Lower bound: $\mathbf{\Omega\left(f^{\frac{1}{2}(1-1/k)}n^{1+1/k} + fn\right)}$ [BDPV '18]
- Upper bound: $\mathbf{O\left(f^{1-1/k} \cdot n^{1+1/k}\right)}$ [BDR '21]

Edge Fault-Tolerant Spanner Bounds

What about edge fault-tolerance?

- Lower bound: $\mathbf{\Omega}\left(\mathbf{f}^{\frac{1}{2}(1-1/k)}\mathbf{n}^{1+1/k} + \mathbf{fn}\right)$ [BDPV '18]
- Upper bound: $\mathbf{O}\left(\mathbf{f}^{1-1/k} \cdot \mathbf{n}^{1+1/k}\right)$ [BDR '21]

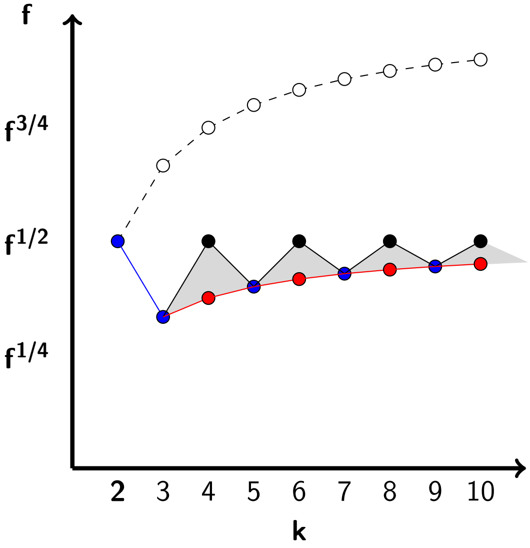
Theorem

Every $\mathbf{n}$*-node graph has an* $\mathbf{f}$*-EFT* $(\mathbf{2k-1})$*-spanner* $\mathbf{H}$ *with*

$$|\mathbf{E}(\mathbf{H})| = \begin{cases} \mathbf{O}\left(\mathbf{k}^2\mathbf{f}^{1/2-1/(2k)}\mathbf{n}^{1+1/k} + \mathbf{kfn}\right) & \mathbf{k} \textit{ is odd} \\ \mathbf{O}\left(\mathbf{k}^2\mathbf{f}^{1/2}\mathbf{n}^{1+1/k} + \mathbf{kfn}\right) & \mathbf{k} \textit{ is even.} \end{cases}$$

Edge Fault-Tolerant Spanner Bounds

dependence on **f**
(**log$_f$** scale)

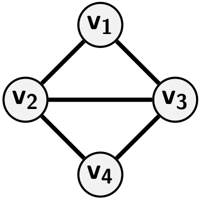


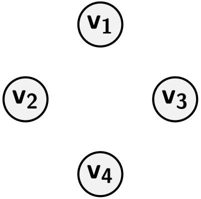
Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

$\mathbf{H} \leftarrow (\mathbf{V}, \varnothing)$
for all $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$ in *nondecreasing* weight order **do**
  if there exists $\mathbf{F} \subseteq \mathbf{V} \setminus \{\mathbf{u}, \mathbf{v}\}$ (for VFT) or $\mathbf{F} \subseteq \mathbf{E}$ (for EFT) with $|\mathbf{H}| \leq \mathbf{f}$ such that $\mathbf{d}_{\mathbf{H} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v}) > (2\mathbf{k} - 1) \cdot \mathbf{w}(\mathbf{u}, \mathbf{v})$ **then**
    add $\{\mathbf{u}, \mathbf{v}\}$ to H
  end if
end for
return H

$\mathbf{k = 2, f = 1}$



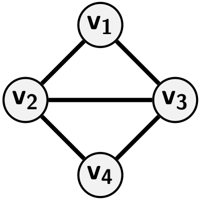


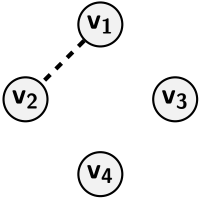
Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

> $\mathbf{H} \leftarrow (\mathbf{V}, \varnothing)$
> **for all** $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$ in *nondecreasing* weight order **do**
>   **if** there exists $\mathbf{F} \subseteq \mathbf{V} \setminus \{\mathbf{u}, \mathbf{v}\}$ (for VFT) or $\mathbf{F} \subseteq \mathbf{E}$ (for EFT) with $|\mathbf{H}| \leq \mathbf{f}$ such that $\mathbf{d}_{\mathbf{H} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v}) > (2\mathbf{k} - 1) \cdot \mathbf{w}(\mathbf{u}, \mathbf{v})$ **then**
>   add $\{\mathbf{u}, \mathbf{v}\}$ to H
>   **end if**
> **end for**
> **return** H

$\mathbf{k = 2, f = 1}$



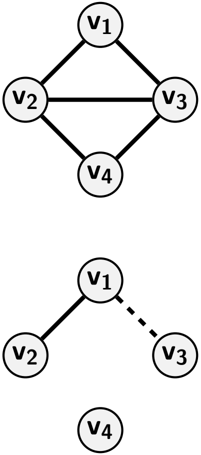


Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

> **$H \leftarrow (V, \varnothing)$**
> **for all** $\{u, v\} \in E$ in *nondecreasing* weight order **do**
>   **if** there exists $F \subseteq V \setminus \{u, v\}$ (for VFT) or $F \subseteq E$ (for EFT) with $|H| \leq f$ such that $d_{H \setminus F}(u, v) > (2k - 1) \cdot w(u, v)$ **then**
>   add $\{u, v\}$ to H
>   **end if**
> **end for**
> **return** H

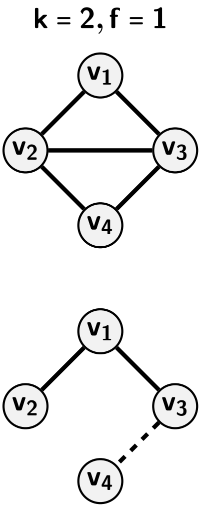
$k = 2, f = 1$



Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

$H \leftarrow (V, \varnothing)$
for all $\{u, v\} \in E$ in *nondecreasing* weight order **do**
  if there exists $F \subseteq V \setminus \{u, v\}$ (for VFT) or $F \subseteq E$ (for EFT) with $|H| \leq f$ such that $d_{H \setminus F}(u, v) > (2k - 1) \cdot w(u, v)$ **then**
    add $\{u, v\}$ to H
  end if
end for
return H

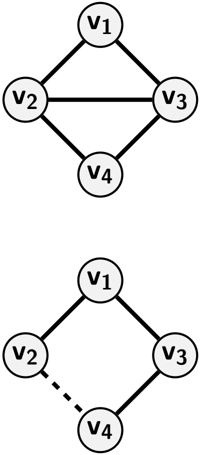


Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

$H \leftarrow (V, \varnothing)$
for all $\{u, v\} \in E$ in *nondecreasing* weight order **do**
  if there exists $F \subseteq V \setminus \{u, v\}$ (for VFT) or $F \subseteq E$ (for EFT) with $|H| \leq f$ such that $d_{H \setminus F}(u, v) > (2k - 1) \cdot w(u, v)$ **then**
    add $\{u, v\}$ to H
  end if
end for
return H

$k = 2, f = 1$

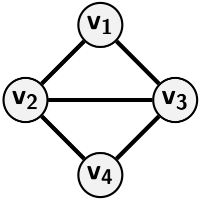


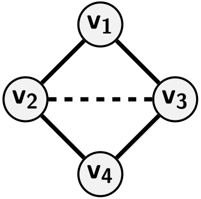
Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

$\mathbf{H} \leftarrow (\mathbf{V}, \varnothing)$
for all $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$ in *nondecreasing* weight order **do**
  if there exists $\mathbf{F} \subseteq \mathbf{V} \setminus \{\mathbf{u}, \mathbf{v}\}$ (for VFT) or $\mathbf{F} \subseteq \mathbf{E}$ (for EFT) with $|\mathbf{H}| \leq \mathbf{f}$ such that $\mathbf{d}_{\mathbf{H} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v}) > (2\mathbf{k} - 1) \cdot \mathbf{w}(\mathbf{u}, \mathbf{v})$ **then**
    add $\{\mathbf{u}, \mathbf{v}\}$ to H
  end if
end for
return H

$\mathbf{k} = 2, \mathbf{f} = 1$



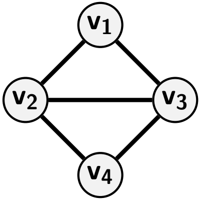


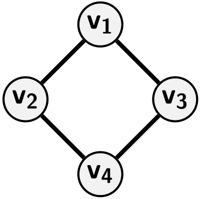
Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

$\mathbf{H} \leftarrow (\mathbf{V}, \varnothing)$
for all $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$ in *nondecreasing* weight order **do**
  if there exists $\mathbf{F} \subseteq \mathbf{V} \setminus \{\mathbf{u}, \mathbf{v}\}$ (for VFT) or $\mathbf{F} \subseteq \mathbf{E}$ (for EFT) with $|\mathbf{H}| \leq \mathbf{f}$ such that $\mathbf{d}_{\mathbf{H} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v}) > (2\mathbf{k} - 1) \cdot \mathbf{w}(\mathbf{u}, \mathbf{v})$ **then**
    add $\{\mathbf{u}, \mathbf{v}\}$ to H
  end if
end for
return H

$\mathbf{k = 2, f = 1}$





Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

> $\mathbf{H} \leftarrow (\mathbf{V}, \varnothing)$
> **for all** $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{E}$ in *nondecreasing* weight order **do**
>   **if** there exists $\mathbf{F} \subseteq \mathbf{V} \setminus \{\mathbf{u}, \mathbf{v}\}$ (for VFT) or $\mathbf{F} \subseteq \mathbf{E}$ (for EFT) with $|\mathbf{H}| \leq \mathbf{f}$ such that $\mathbf{d}_{\mathbf{H} \setminus \mathbf{F}}(\mathbf{u}, \mathbf{v}) > (\mathbf{2k} - \mathbf{1}) \cdot \mathbf{w}(\mathbf{u}, \mathbf{v})$ **then**
>   add $\{\mathbf{u}, \mathbf{v}\}$ to H
>   **end if**
> **end for**
> **return** H

Intuition: H should be "almost" high-girth

- How do we define "almost"?

Greedy Fault-Tolerant Spanner Algorithm

Originally proposed by [Bodwin, D, Parter, Vassilevska Williams '18]

> **$H \leftarrow (V, \varnothing)$**
> **for all** $\{u, v\} \in E$ in *nondecreasing* weight order **do**
>   **if** there exists $F \subseteq V \setminus \{u, v\}$ (for VFT) or $F \subseteq E$ (for EFT) with $|H| \leq f$ such that $d_{H \setminus F}(u, v) > (2k - 1) \cdot w(u, v)$ **then**
>   add $\{u, v\}$ to H
>   **end if**
> **end for**
> **return** H

Intuition: **H** should be "almost" high-girth

- How do we define "almost"?

Structural: There is a large high-girth subgraph in **H**

Moore-like: Suitable adaptations of the arguments for high-girth graphs also apply

Main Difficulty for Edge Fault-Tolerance

VFT: structural approach (BP '19, BDR '21)

- Greedy spanner "almost" high-girth because it has large high-girth subgraph
  - Blocking sets (BP'19), or direct from algorithm (BDR '21)

Problem: Can't use this idea to get *improved* bounds for edge fault-tolerance!

- Bodwin-Patel showed can't use blocking sets to get below $\mathbf{f^{1-1/k} n^{1+1/k}}$
- (this paper, informal): if there was a structural argument, then it would imply the Erdős girth conjecture for $\mathbf{k} = \mathbf{7}$ (currently unknown)

Main Difficulty for Edge Fault-Tolerance

VFT: structural approach (BP '19, BDR '21)

- Greedy spanner "almost" high-girth because it has large high-girth subgraph
  - Blocking sets (BP'19), or direct from algorithm (BDR '21)

Problem: Can't use this idea to get *improved* bounds for edge fault-tolerance!

- Bodwin-Patel showed can't use blocking sets to get below $\mathbf{f^{1-1/k} n^{1+1/k}}$
- (this paper, informal): if there was a structural argument, then it would imply the Erdős girth conjecture for $\mathbf{k} = \mathbf{7}$ (currently unknown)

Approach:

- *Strong* blocking sets
- More sophisticated version of Moore bounds on graphs with small strong blocking sets

Strong Blocking Sets

Definition (strong $\mathbf{t}$-blocking set)

A *strong* $\mathbf{t}$-*blocking set* of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a set $\mathbf{B} \subseteq \mathbf{E} \times \mathbf{E}$ where for every cycle $\mathbf{C}$ in $\mathbf{G}$ with $|\mathbf{C}| \leq \mathbf{t}$, there exists $(\mathbf{e}, \mathbf{e}') \in \mathbf{B}$ such that:

- $\mathbf{e}, \mathbf{e}' \in \mathbf{C}$ and $\mathbf{e} \neq \mathbf{e}'$, and
- Either $\mathbf{e}$ or $\mathbf{e}'$ is the *highest-weight* edge in $\mathbf{C}$

If $\mathbf{G}$ unweighted, "highest-weight" determined by ordering used by greedy algorithm.

Strong Blocking Sets

Definition (strong $\mathbf{t}$-blocking set)

A *strong* $\mathbf{t}$*-blocking set* of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a set $\mathbf{B} \subseteq \mathbf{E} \times \mathbf{E}$ where for every cycle $\mathbf{C}$ in $\mathbf{G}$ with $|\mathbf{C}| \leq \mathbf{t}$, there exists $(\mathbf{e}, \mathbf{e}') \in \mathbf{B}$ such that:

- $\mathbf{e}, \mathbf{e}' \in \mathbf{C}$ and $\mathbf{e} \neq \mathbf{e}'$, and
- Either $\mathbf{e}$ or $\mathbf{e}'$ is the *highest-weight* edge in $\mathbf{C}$

If $\mathbf{G}$ unweighted, "highest-weight" determined by ordering used by greedy algorithm.

Lemma

The subgraph $\mathbf{H}$ *output by the greedy algorithm has a strong* $\mathbf{2k}$*-blocking set of size at most* $\mathbf{f}|\mathbf{E}(\mathbf{H})|$.

Proof sketch: Same as non-strong lemma from Bodwin-Patel '19

Moore Bounds

Can't use structural approach – use strong blocking set for modified Moore bounds.

Theorem (Moore Bounds)

Any graph $\mathbf{G}$ *with girth at least* $\mathbf{2k+1}$ *has at most* $\mathbf{O(n^{1+1/k})}$ *edges*

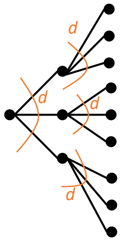
Moore Bounds

Can't use structural approach – use strong blocking set for modified Moore bounds.

Theorem (Moore Bounds)

Any graph $\mathbf{G}$ *with girth at least* $\mathbf{2k+1}$ *has at most* $\mathbf{O(n^{1+1/k})}$ *edges*

Counting Lemma: Let $\mathbf{d}$ be average degree of $\mathbf{G}$. Then $\mathbf{G}$ has at least $\mathbf{\Omega(n \cdot d^k)}$ simple $\mathbf{k}$-paths



Moore Bounds

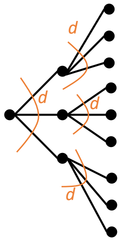
Can't use structural approach – use strong blocking set for modified Moore bounds.

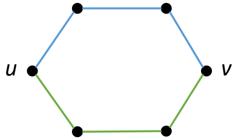
Theorem (Moore Bounds)

Any graph $\mathbf{G}$ *with girth at least* $\mathbf{2k+1}$ *has at most* $\mathbf{O(n^{1+1/k})}$ *edges*

Counting Lemma: Let $\mathbf{d}$ be average degree of $\mathbf{G}$. Then $\mathbf{G}$ has at least $\mathbf{\Omega(n \cdot d^k)}$ simple $\mathbf{k}$-paths

Dispersion Lemma: No two simple $\mathbf{k}$-paths can share the same endpoints $\implies \leq \mathbf{n^2}$ simple $\mathbf{k}$-paths





Moore Bounds

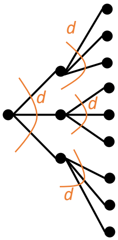
Can't use structural approach – use strong blocking set for modified Moore bounds.

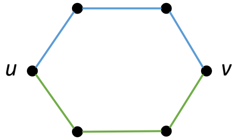
Theorem (Moore Bounds)

Any graph $\mathbf{G}$ *with girth at least* $\mathbf{2k+1}$ *has at most* $\mathbf{O(n^{1+1/k})}$ *edges*

Counting Lemma: Let $\mathbf{d}$ be average degree of $\mathbf{G}$. Then $\mathbf{G}$ has at least $\mathbf{\Omega(n \cdot d^k)}$ simple $\mathbf{k}$-paths

Dispersion Lemma: No two simple $\mathbf{k}$-paths can share the same endpoints $\implies \leq \mathbf{n^2}$ simple $\mathbf{k}$-paths





$$\mathbf{n \cdot d^k \leq n^2 \implies d \leq n^{1/k} \implies |E| = nd/2 = O(n^{1+1/k})}$$

Generalized Dispersion Lemma

Greedy spanner **H** has small strong blocking set.

Generalized Dispersion Lemma

Greedy spanner $\mathbf{H}$ has small strong blocking set.

$\implies$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, there is some set $\mathbf{S_{uv}}$ of $\mathbf{O(kf)}$ edges such that all simple $\mathbf{u} - \mathbf{v}$ $\mathbf{k}$-paths use some edge of $\mathbf{S_{uv}}$ as *heaviest* edge.

Generalized Dispersion Lemma

Greedy spanner $\mathbf{H}$ has small strong blocking set.

$\implies$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, there is some set $\mathbf{S_{uv}}$ of $\mathbf{O(kf)}$ edges such that all simple $\mathbf{u}-\mathbf{v}$ $\mathbf{k}$-paths use some edge of $\mathbf{S_{uv}}$ as *heaviest* edge.

Idea: For each $\mathbf{(x, y)} \in \mathbf{S_{uv}}$, inductively bound $\#$ $\mathbf{u}-\mathbf{x}$ paths and $\#$ $\mathbf{y}-\mathbf{v}$ paths

Generalized Dispersion Lemma

Greedy spanner $\mathbf{H}$ has small strong blocking set.

$\implies$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, there is some set $\mathbf{S_{uv}}$ of $\mathbf{O(kf)}$ edges such that all simple $\mathbf{u-v}$ $\mathbf{k}$-paths use some edge of $\mathbf{S_{uv}}$ as *heaviest* edge.

Idea: For each $\mathbf{(x, y)} \in \mathbf{S_{uv}}$, inductively bound # $\mathbf{u-x}$ paths and # $\mathbf{y-v}$ paths

To make induction work, helpful if heaviest edge *not* first or last, and for this to be true throughout induction

- Only count simple *alternating* $\mathbf{k}$-paths: each even hop heavier than adjacent (odd) hops

Generalized Dispersion Lemma

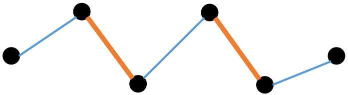
Greedy spanner $\mathbf{H}$ has small strong blocking set.

$\Longrightarrow$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, there is some set $\mathbf{S_{uv}}$ of $\mathbf{O(kf)}$ edges such that all simple $\mathbf{u} - \mathbf{v}$ $\mathbf{k}$-paths use some edge of $\mathbf{S_{uv}}$ as *heaviest* edge.

Idea: For each $(\mathbf{x}, \mathbf{y}) \in \mathbf{S_{uv}}$, inductively bound $\#$ $\mathbf{u} - \mathbf{x}$ paths and $\#$ $\mathbf{y} - \mathbf{v}$ paths

To make induction work, helpful if heaviest edge *not* first or last, and for this to be true throughout induction

- Only count simple *alternating* $\mathbf{k}$-paths: each even hop heavier than adjacent (odd) hops



Generalized Dispersion Lemma

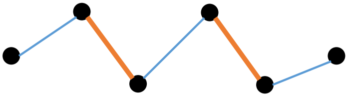
Greedy spanner $\mathbf{H}$ has small strong blocking set.

$\Longrightarrow$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, there is some set $\mathbf{S_{uv}}$ of $\mathbf{O(kf)}$ edges such that all simple $\mathbf{u-v}$ $\mathbf{k}$-paths use some edge of $\mathbf{S_{uv}}$ as *heaviest* edge.

Idea: For each $(\mathbf{x}, \mathbf{y}) \in \mathbf{S_{uv}}$, inductively bound # $\mathbf{u-x}$ paths and # $\mathbf{y-v}$ paths

To make induction work, helpful if heaviest edge *not* first or last, and for this to be true throughout induction

- Only count simple *alternating* $\mathbf{k}$-paths: each even hop heavier than adjacent (odd) hops



Lemma (Generalized Dispersion)

For any nodes $\mathbf{u}, \mathbf{v}$, the number of simple alternating $\mathbf{u-v}$ $\mathbf{k}$-paths is

$$\mathbf{O}\left(\mathbf{k}^2\mathbf{f}\right)^{(\mathbf{k}-1)/2} \qquad \mathbf{k} \text{ is odd}$$

$$\mathbf{O}\left(\mathbf{k}^2\mathbf{f}\right)^{\mathbf{k}/2} \qquad \mathbf{k} \text{ is even}$$

Generalized Counting Lemma

Need to show there are many simple alternating **k**-paths.

Generalized Counting Lemma

Need to show there are many simple alternating **k**-paths.

Lemma: Any graph with at least **kn** edges has at least one simple alternating **k**-path

Proof: Induction on **k**

Generalized Counting Lemma

Need to show there are many simple alternating **k**-paths.

Lemma: Any graph with at least **kn** edges has at least one simple alternating **k**-path

Proof: Induction on **k**

Lemma: Any graph with at least **2kn** edges has at least **kn** simple alternating **k**-paths

Proof: Find a distinct simple alternating **k**-path for each "extra" edge

Generalized Counting Lemma

Need to show there are many simple alternating **k**-paths.

Lemma: Any graph with at least **kn** edges has at least one simple alternating **k**-path

Proof: Induction on **k**

Lemma: Any graph with at least **2kn** edges has at least **kn** simple alternating **k**-paths

Proof: Find a distinct simple alternating **k**-path for each "extra" edge

Lemma (Generalized Counting)

H *has* $\mathbf{\Omega(n \cdot (d/k)^k)}$ *simple alternating* **k**-*paths.*

Generalized Counting Lemma

Need to show there are many simple alternating $\mathbf{k}$-paths.

Lemma: Any graph with at least $\mathbf{kn}$ edges has at least one simple alternating $\mathbf{k}$-path

Proof: Induction on $\mathbf{k}$

Lemma: Any graph with at least $\mathbf{2kn}$ edges has at least $\mathbf{kn}$ simple alternating $\mathbf{k}$-paths

Proof: Find a distinct simple alternating $\mathbf{k}$-path for each "extra" edge

Lemma (Generalized Counting)

$\mathbf{H}$ *has* $\mathbf{\Omega(n \cdot (d/k)^k)}$ *simple alternating* $\mathbf{k}$*-paths.*

Proof Sketch.

Sample nodes to get subgraph, use previous lemma to argue many simple alternating $\mathbf{k}$-paths, scale back up. □

Putting It Together

Count simple alternating **k**-paths.

Odd **k**:

$$\Omega(n \cdot (d/k)^k) = O\left(n^2 \left(k^2 f\right)^{(k-1)/2}\right)$$

$$\implies d/k = O\left(n^{1/k} \left(k^2 f\right)^{\frac{1}{2}(1-1/k)}\right)$$

$$\implies |E(H)| = \frac{nd}{2} = O\left(k^2 f^{\frac{1}{2}(1-1/k)} n^{1+1/k}\right)$$

Even **k**:

$$\Omega(n \cdot (d/k)^k) = O\left(n^2 \left(k^2 f\right)^{k/2}\right)$$

$$\implies d/k = O\left(n^{1/k} \left(k^2 f\right)^{1/2}\right)$$

$$\implies |E(H)| = \frac{nd}{2} = O\left(k^2 f^{1/2} n^{1+1/k}\right)$$

Final Notes

Optimal for odd constant $\mathbf{k}$, off by $\mathbf{f^{1/(2k)}}$ for even constant $\mathbf{k}$.

Main open question: close gap for even $\mathbf{k}$!

- Essentially always see difference between even/odd *stretch* due to bipartiteness (hence why stretch is always $\mathbf{2k-1}$)
- Rare (but not unheard of) to see difference between even/odd $\mathbf{k}$.
- What is the correct bound???

Also off by $\mathbf{k^2}$, but WLOG $\mathbf{k \le O(\log n)}$. Still would like to get rid of $\mathbf{k}$ factors!

Algorithm as stated takes exponential time!

- Can turn into polytime using same idea as [D, Robelle PODC '20]. Extra loss of $\mathbf{O(k^{1/2})}$

Thanks!