Distributed Minimum Degree Spanning Trees

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Minimum Degree Spanning Tree

Input: Graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
Feasible Solution: Spanning tree $\mathbf{T} = (\mathbf{V}, \mathbf{E}')$
Objective: Minimize max degree $\mathbf{\Delta}(\mathbf{T})$ of $\mathbf{T}$

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- Classical NP-hard problem (reduction from Hamiltonian Path)
- Natural in distributed/networked settings:
  - Building a low-degree backbone network
  - Broadcast capacity in mobile telephone model [D, Halldórsson, Newport, Weaver DISC '19]
  - In many networking scenarios, degree ≈ load. Min max load.
  - Lots of attention to MST problem – why not MDST?

Algorithms for MDST

- Centralized ($\mathbf{d = OPT}$):
  - [Fürer, Raghavachari SODA '92]: Polytime $\mathbf{(d+1)}$-solution via complex recursive local search (*semi-local improvements*)
  - Simpler version (non-recursive) gives $\mathbf{(2d + \log n)}$-solution (*local improvements*)

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- Distributed:
  - [Blin, Butelle IPDPS '03]: Each local and semi-local improvement of FR can be computed in a distributed way
    - But separate improvements not computed in parallel, so still large running time ($\mathbf{\Omega(n)}$)
  - Self-stabilizing algorithms [Blin, Fraigniaud ICDCS '15], [Blin, Potop-Butucaru, Rovedakis '11]: running times $\mathbf{\Omega(n^2)}$
  - More general problem: find MST which minimizes max degree.
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Question: Can we provide good approximations for MDST in time $\mathbf{O(D+\sqrt{n})}$ (like for MST)?

Our Results: Upper Bounds

Models:

- CONGEST: synchronous rounds, each message size at most $\mathbf{O(\log n)}$ bits ($\mathbf{O(1)}$ words)
- broadcast-CONGEST: in each round, each node sends *same* message to all neighbors

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Theorem

There is a randomized algorithm in the broadcast-CONGEST model which builds a spanning tree of maximum degree at most $\mathbf{4(1+\epsilon)d + O(\log n)}$ *and has expected running time at most* $\mathbf{O((D+\sqrt{n})\log^4 n)}$

Theorem

There is a deterministic algorithm in the CONGEST model which builds a spanning tree of maximum degree at most $\mathbf{4(1+\epsilon)d + O(\log n)}$ *and has expected running time at most* $\mathbf{O((D+\sqrt{n})\log^5 n)}$

Our Results: Lower Bounds

- Want to show that polynomial dependence on n is necessary
- Precise lower bound rather complex. Some simple corollaries:

Theorem

For any $\epsilon < 1/6$, there exists a family of instances of diameter $D = \Theta(n^{1/2-3\epsilon} + \log n)$ where any MDST algorithm with a polylogarithmic multiplicative approximation factor needs $\tilde{\Omega}(n^{1/2-\epsilon} + D)$ rounds.

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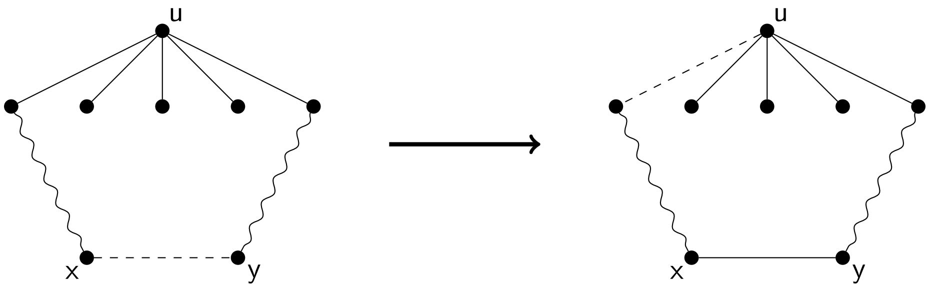
Today: only upper bounds

Local improvements

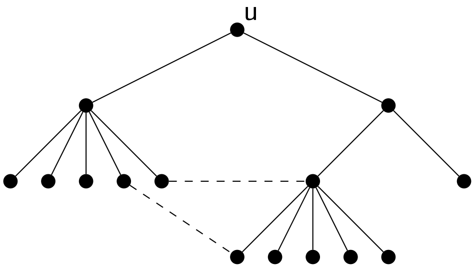
Idea: Adding an edge creates a cycle. Can remove any edge in that cycle.



- Degrees of $\mathbf{x}, \mathbf{y}$ increase by $\mathbf{1}$, but degree of $\mathbf{u}$ decreases by $\mathbf{1}$.
- FR: Make local improvements to decrease degrees of high-degree nodes, until no such local improvements exist.

Parallel Improvements

To get $\mathbf{o(n)}$ running time, need to do many improvements in parallel. Requires coordination – not all local improvements can be done simultaneously!



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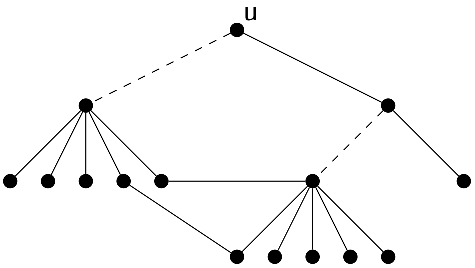

Everything else

- **Def:** Let $\mathbf{X_k}$ be nodes of degree at least $\mathbf{k}$ in current tree $\mathbf{T}$.
- Two parameters
  - γ: definition of "high-degree" (nodes in $\mathbf{X}_\gamma$)
  - $\mathbf{q}$: definition of "low-degree" (degree at most $\gamma_0 := \gamma - 2\mathbf{q}$)
- Local improvements:
  - Hurt low-degree nodes by at most $\mathbf{q}$
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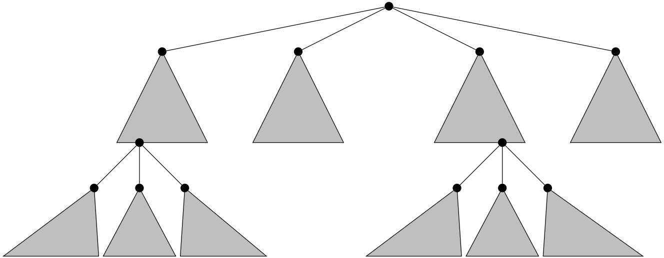
Main point (rest of talk): want to find lots of parallel local improvements.

Leaf Branches

- *Branch*: Component of $\mathbf{T} \setminus \mathbf{X}_\gamma$
- *Leaf branch*: branch $\mathbf{C}$ where there is only one edge in $\mathbf{T}$ leaving $\mathbf{C}$.



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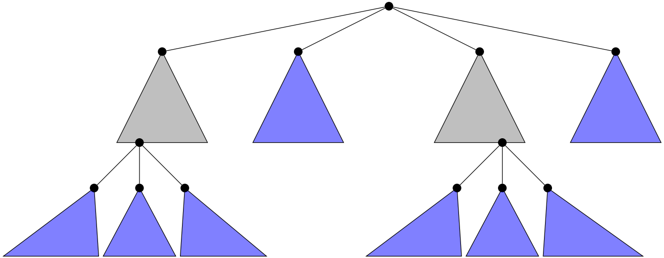

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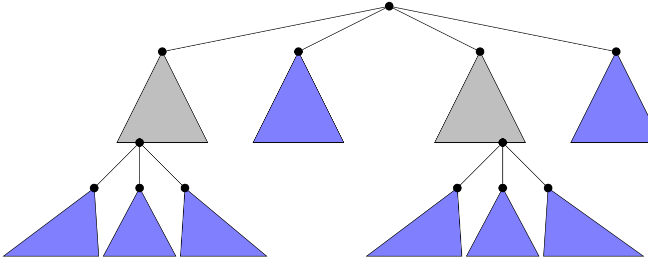

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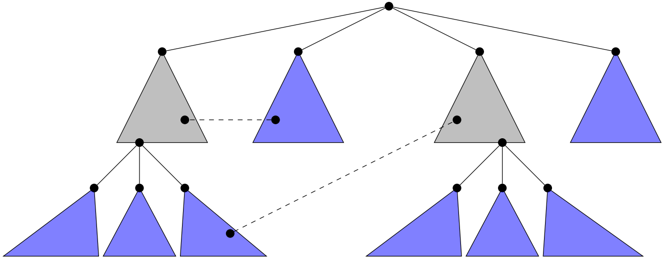

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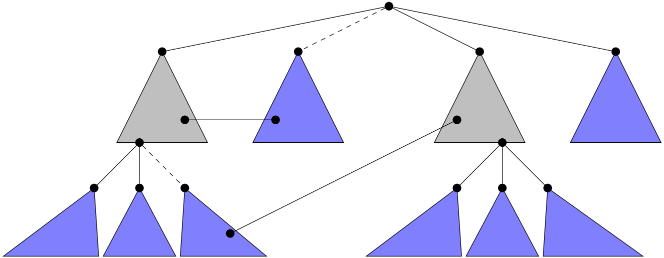

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Improvement Graph

- Edge is *good* if:
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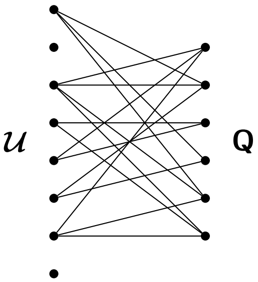

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- $\mathbf{Q}$: nodes of degree $\leq \gamma_0$
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Approach: Find a large $(\mathbf{1}, \mathbf{q})$-matching in improvement graph (degree $\mathbf{1}$ on leaf branches, degree $\leq \mathbf{q}$ on low-degree nodes).

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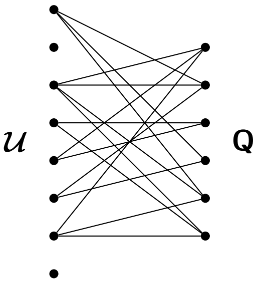

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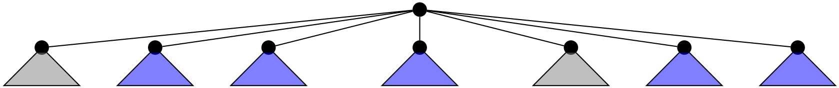
- What about requirement that high-degree edges improve at most $\mathbf{q}$?

Constrained **q**-Matching

High-degree node improves ≥ **q** only if ≥ **q** leaf branches adjacent to it are matched.



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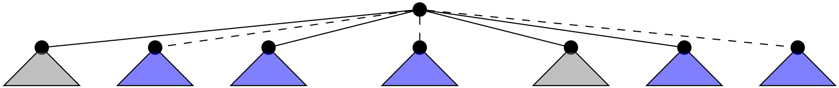

Constrained **q**-*Matching*: force this not to happen

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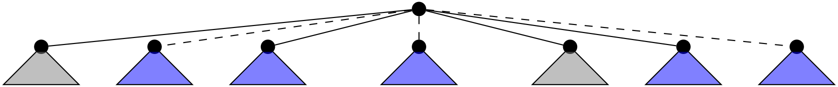

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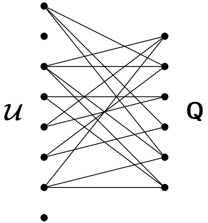

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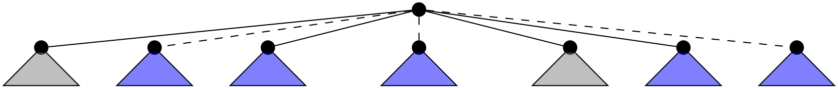

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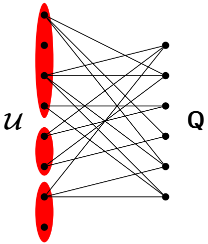

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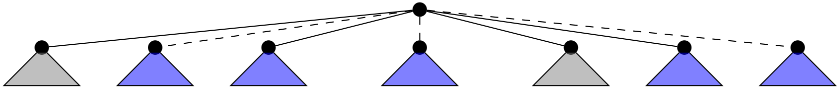

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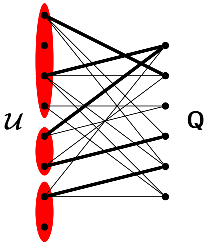

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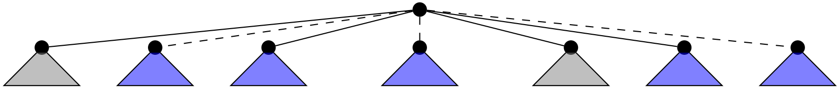

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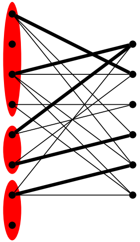

So want to prove:

- There must exist a large Constrained **q**-Matching (if γ large enough and not many "medium-degree" nodes)
- Can quickly find $\mathbf{O(1)}$-approximation to Max Constrained **q**-Matching.

Large Constrained $\mathbf{q}$-Matching

Theorem

There is a constrained $\mathbf{q}$*-matching of size at least* $\frac{\mathbf{q}}{\gamma}((\gamma - \mathbf{2})|\mathbf{X}_\gamma| - \mathbf{d}|\mathbf{X}_{\gamma_0}|)$ *(where* $\mathbf{d}$ *is optimal max degree)*

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Proof Sketch:

- Each node in $\mathbf{X}_\gamma$ causes lots of leaf branches when removed.
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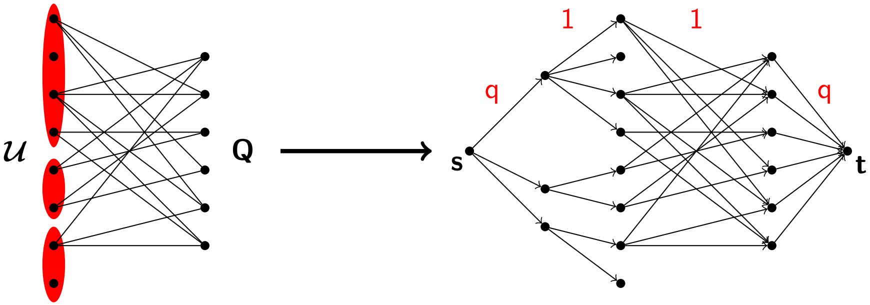
So if $|\mathbf{X}_{\gamma_0}|$ not much larger than $\mathbf{X}_\gamma$ and γ large enough, about $\mathbf{q}|\mathbf{X}_\gamma|$

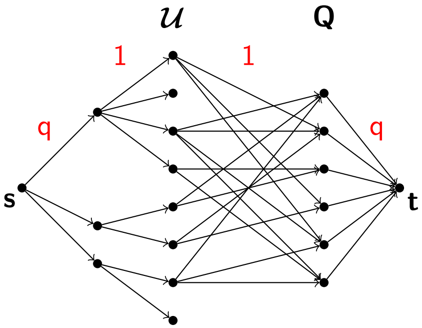
Approximating Constrained **q**-Matching

Turn constrained **q**-matching into flow:



- Integral capacities, so integral max flow
- Flow of α iff constrained **q**-matching of size α

Approximate Max Flow



Lemma: In depth-**d** flow network, any maximal flow has value at least **(1/d)** of max flow.

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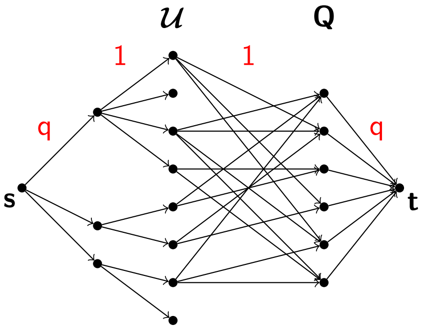

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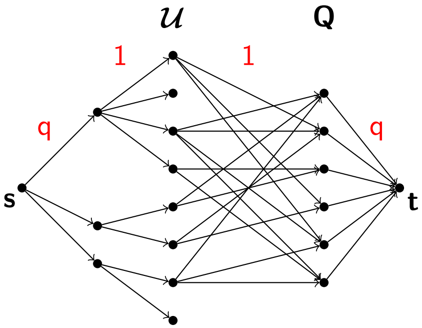

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- Randomized rounding to get approximate constrained **q**-matching
- Some annoying details ($\mathcal{U}$ are actually components, not nodes)

Putting it Together

- Want to find many parallel local improvements
- Restrict to "safe" improvements: add good edge, remove edge from leaf branch
- Prove there have to be many safe improvements (or else finished)
- Algorithm for approximating max safe improvements via max flow, randomized rounding.
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- Approximation guarantee:
  - Somewhat complex/delicate
  - Morally: same as centralized FR, but with small extra loss (2 to 4 multiplicative)

Conclusion

Results:

- Gave first $\tilde{\mathbf{O}}(\mathbf{D}+\sqrt{\mathbf{n}})$-round $\mathbf{O}(\mathbf{d}+\log \mathbf{n})$-degree bounded MDST algorithm in CONGEST
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Open Problems:

- Key idea was finding many local improvements which can be done in parallel. Should imply PRAM, streaming, etc. algorithms?
- Stronger approximation! Ideally: degree $d + 1$?
- Steiner tree instead of spanning?
  - Centralized bounds same (degree $d + 1$ from local search)
  - Centralized algorithm makes local improvements using *paths* instead of edges. Distributed?

Thanks!