Load Balancing with Bounded Convergence in Dynamic Networks

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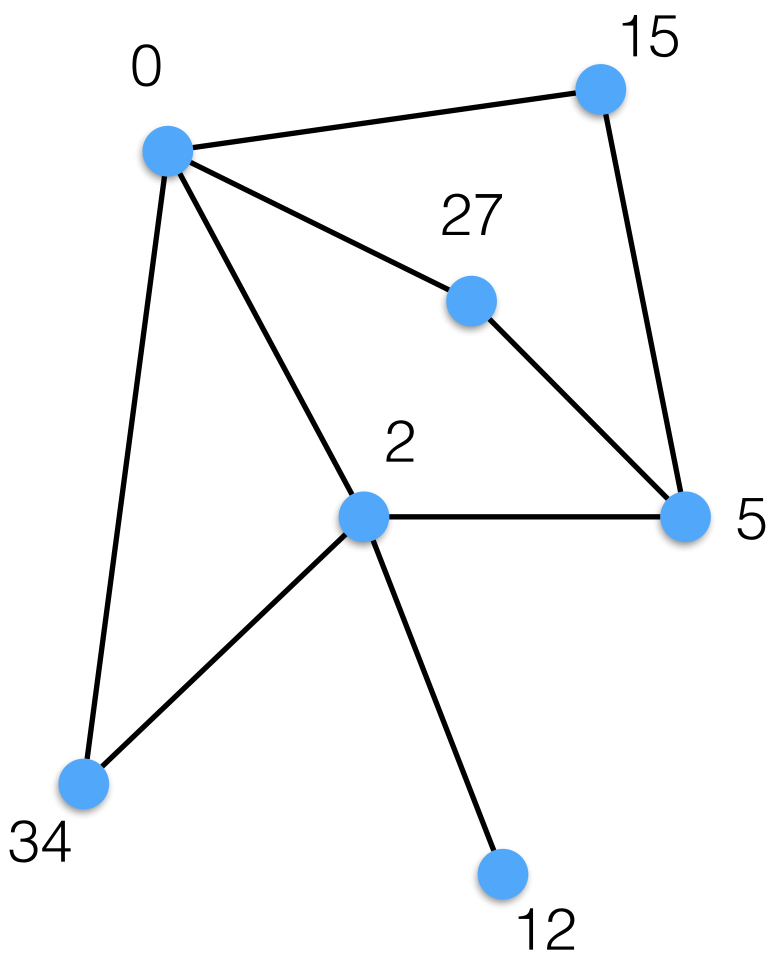
Load Balancing in Graphs



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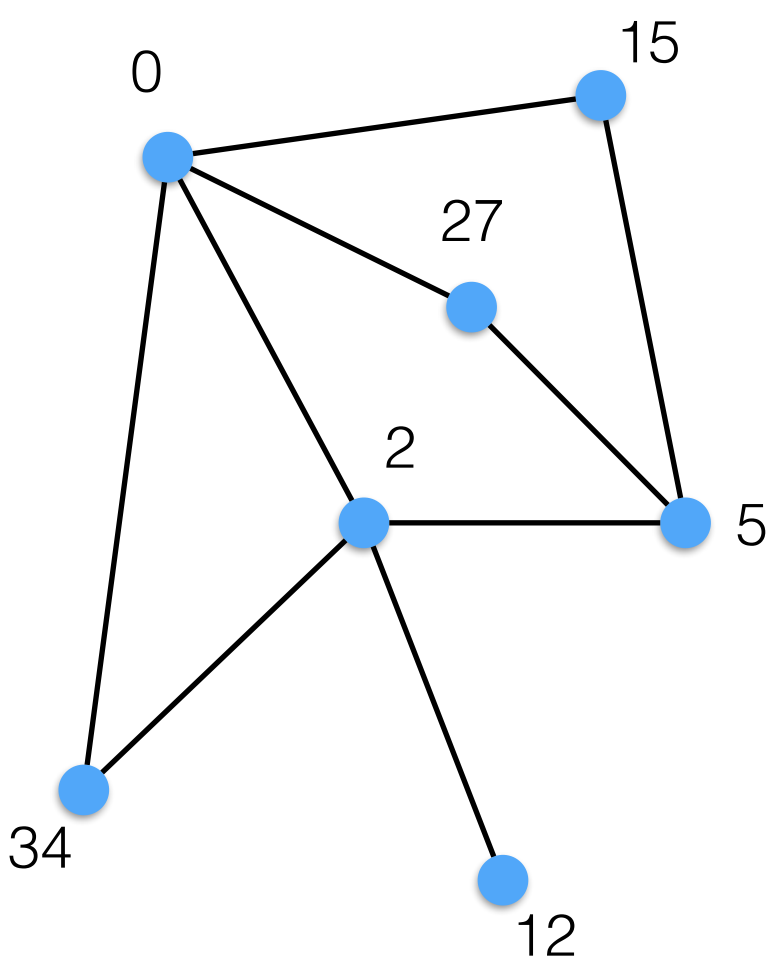

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  - $T = \sum_u x_u$
- Goal: redistribute work so that every node has approximately T/n load

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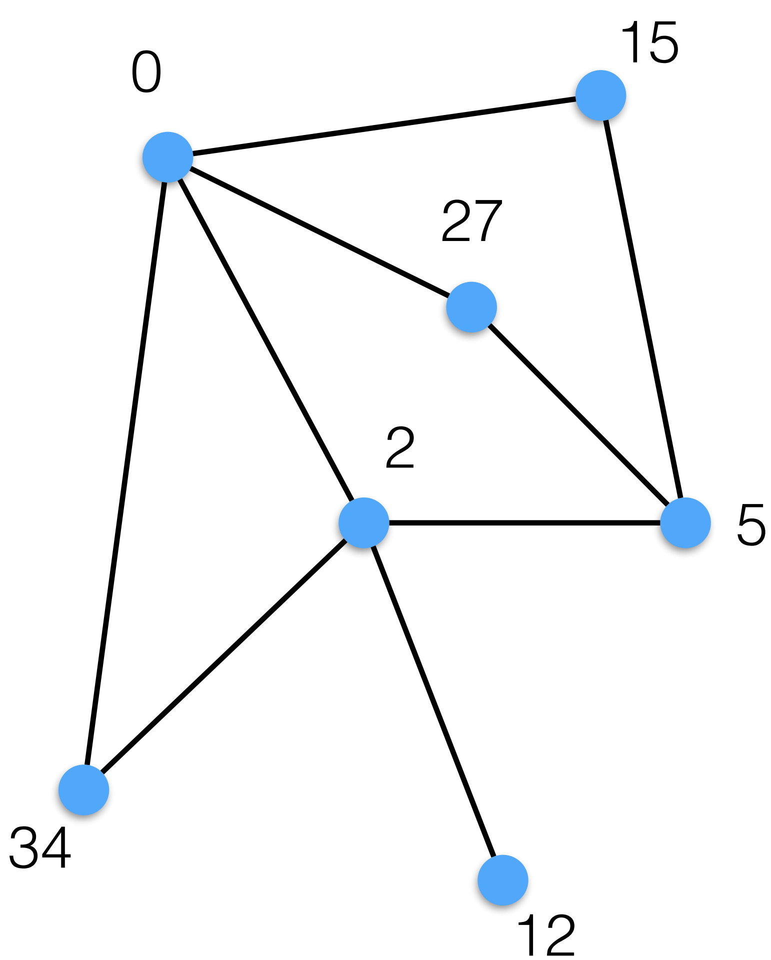

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- Goal: redistribute work so that every node has approximately T/n load
- Can only send work along edges of graph, no global knowledge (Local Load Balancing)
- Synchronous rounds: transfer work along edges in each round

Desirable Properties

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- In each round, edges used form a matching
  - Each node sends/receives work from at most one other node in each round
  - Important for some applications/models [Cybenko '89]
  - Dimension Exchange
- Works in dynamic graphs
  - Sequence of graphs $H = (G_1 = (V, E_1), G_2 = (V, E_2), \ldots)$, each connected
  - Distributed: each node sees only load of neighbors in current graph
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- Converges *quickly*
- Many results getting 2/3 — can we get all three?

Model

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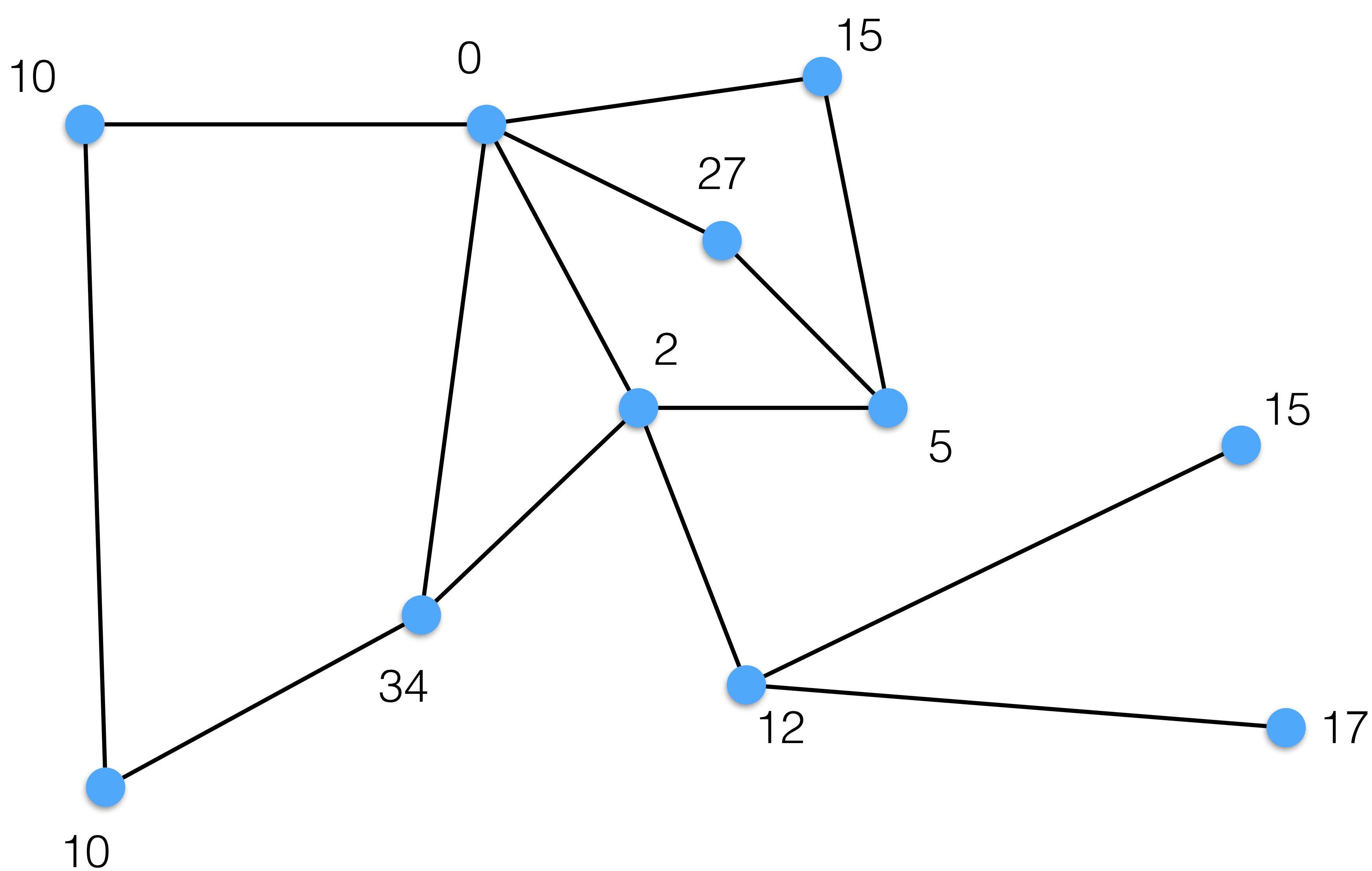
- Goal: τ-convergence
  - $|x_u(u) - x_v(r)| \leq \tau$ for all $u,v \in V$

Example



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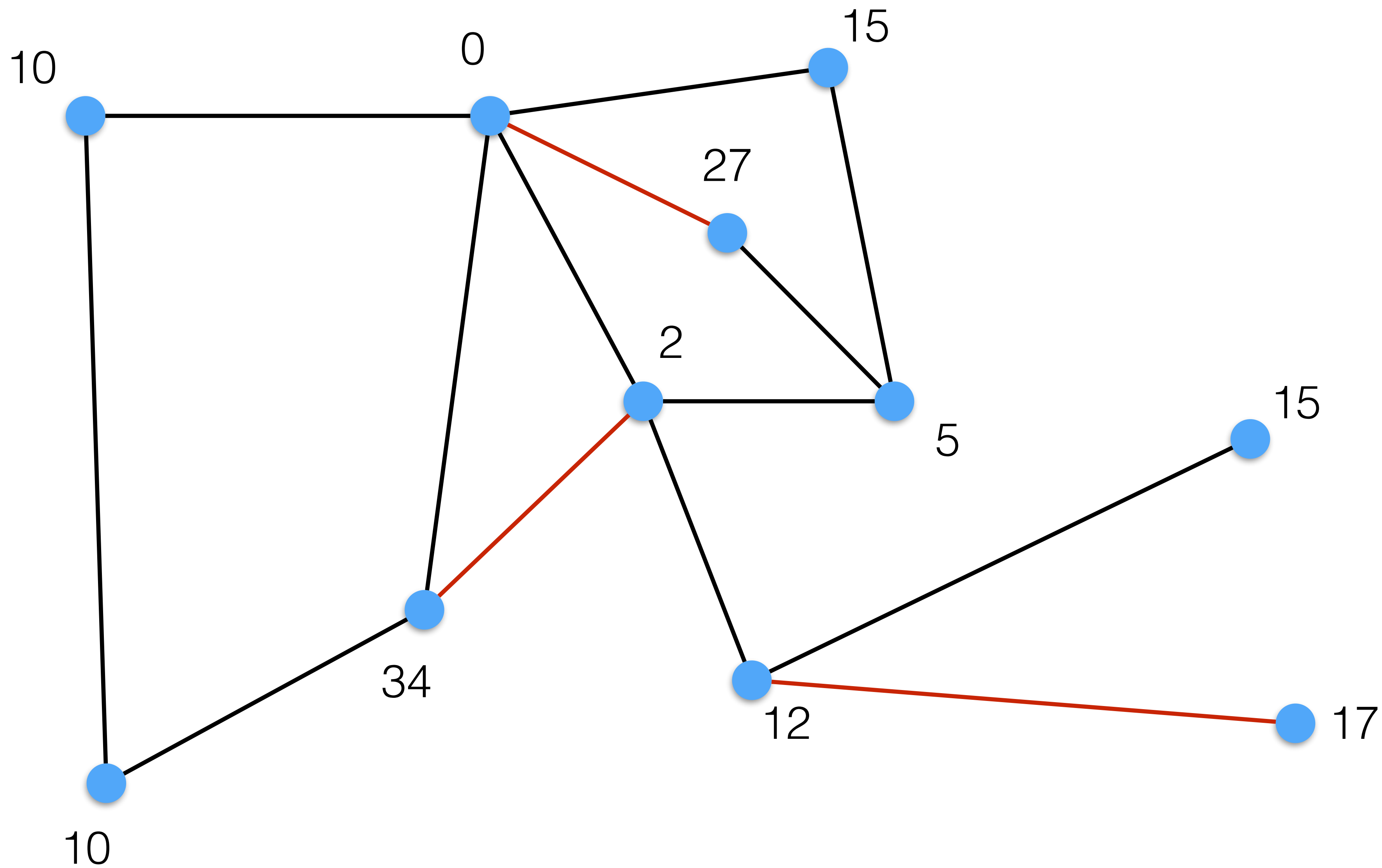

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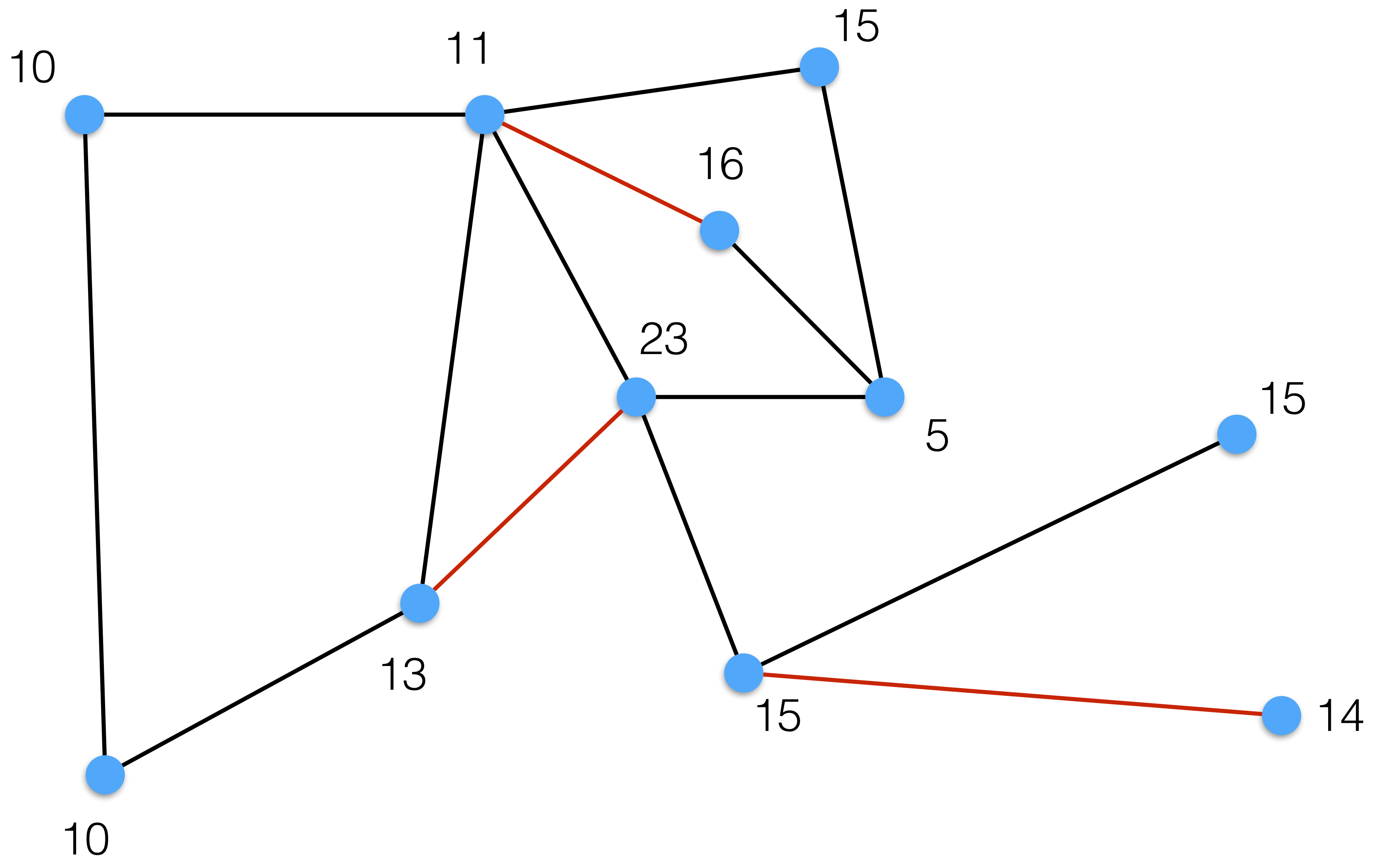

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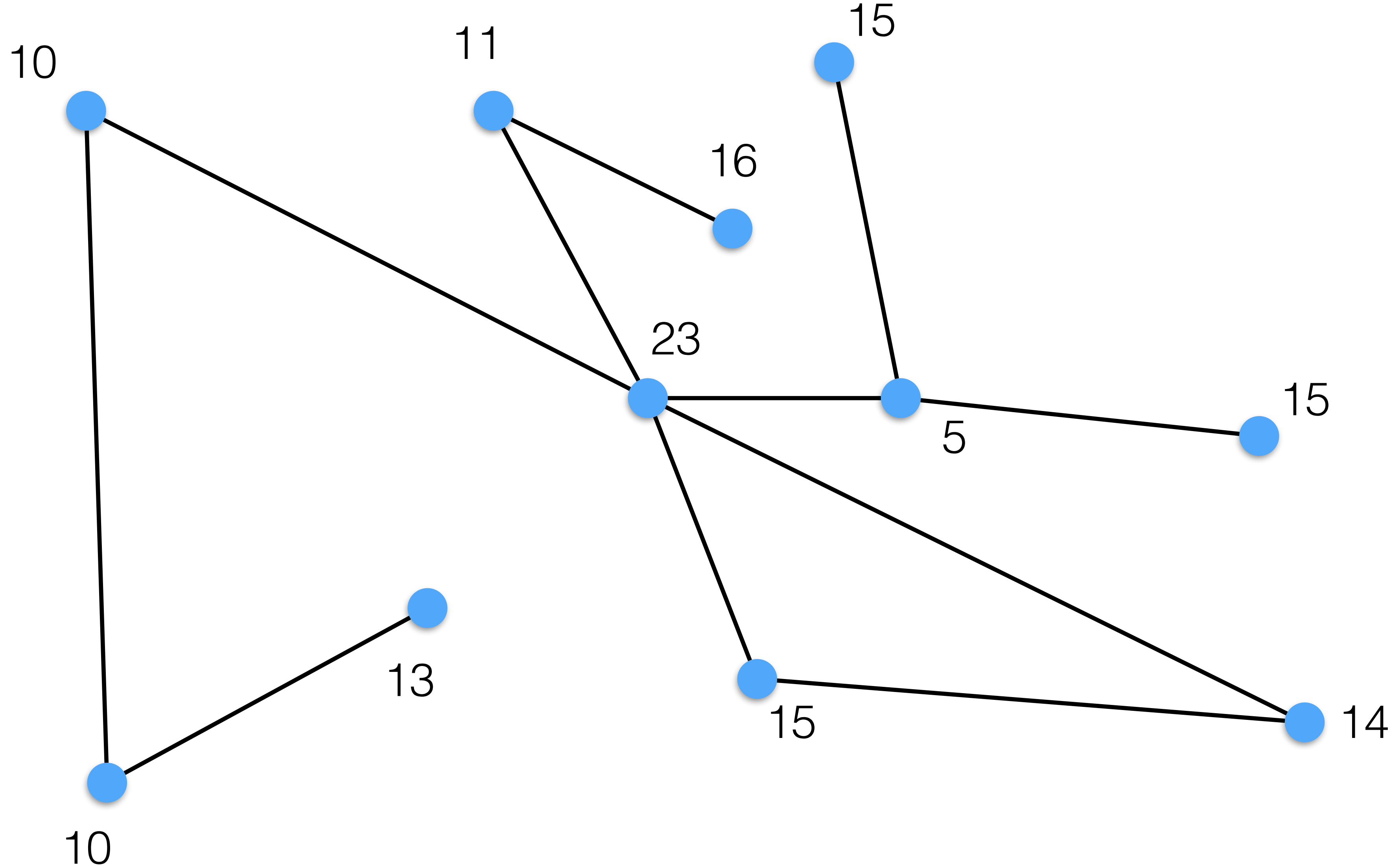

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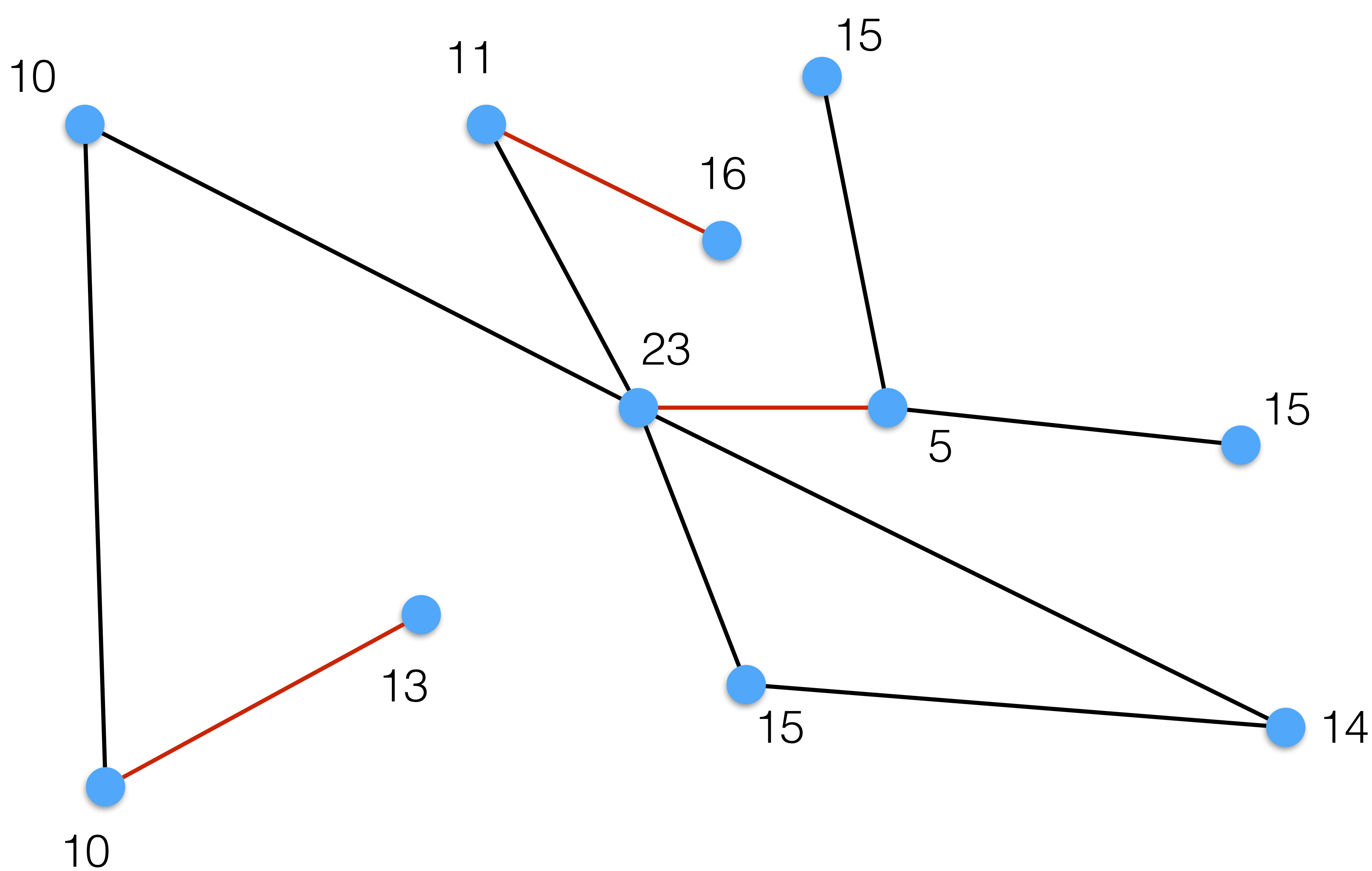

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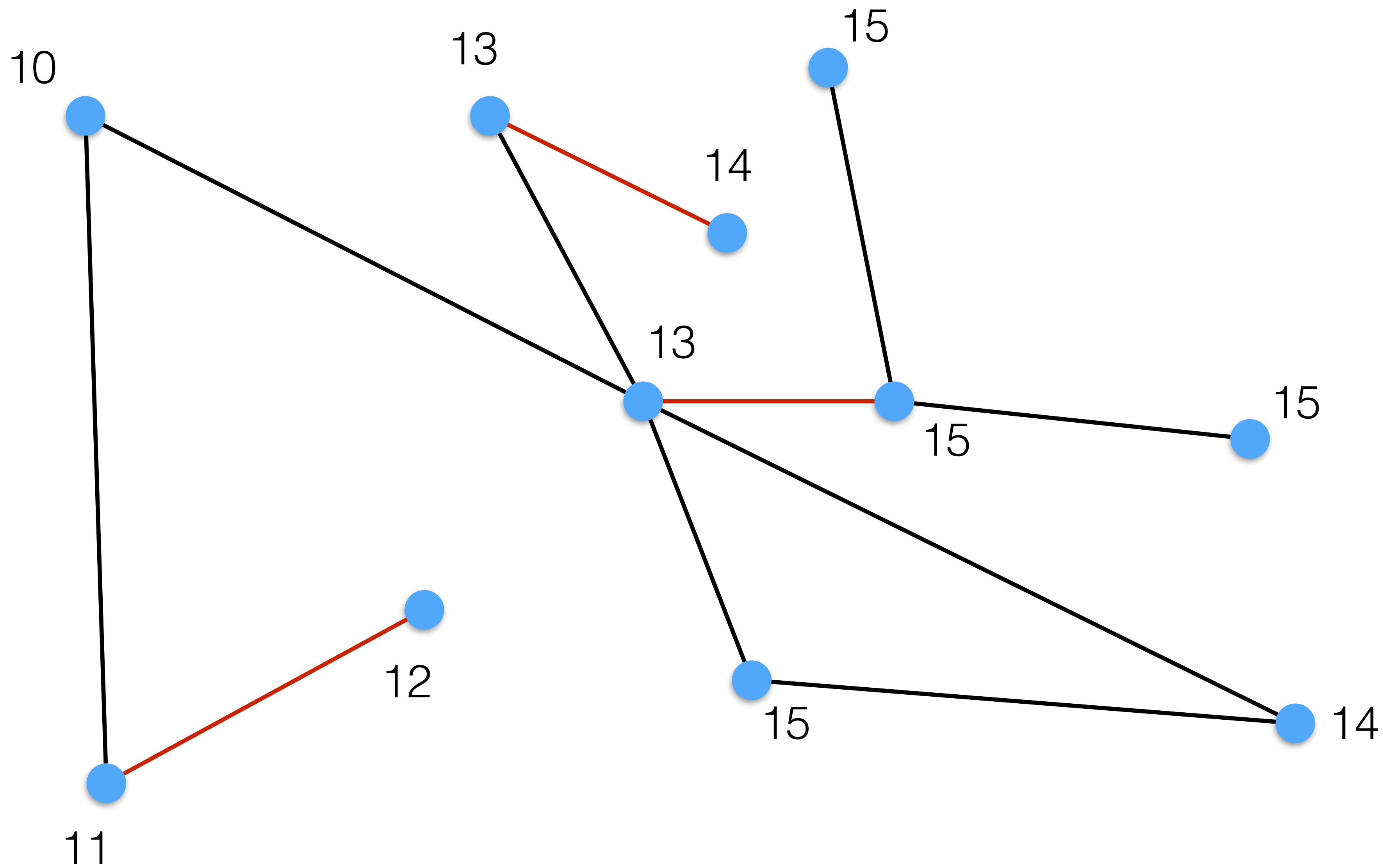

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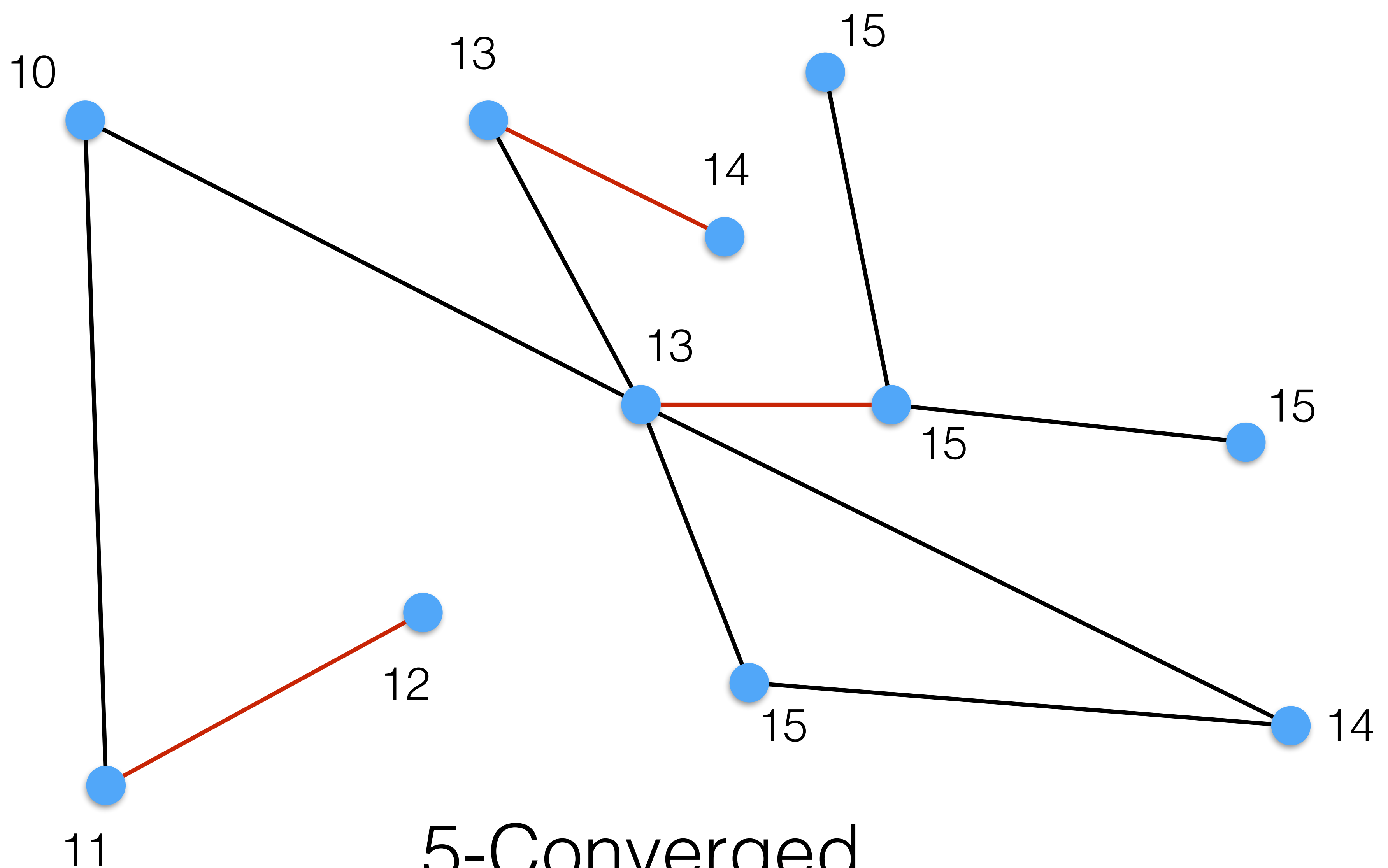

5-Converged

Results: Upper Bound

Theorem: There is an algorithm which achieve τ-convergence after

$$O\left(\min\left(n^2\log\left(\frac{Tn}{\tau}\right),\frac{Tn\log n}{\tau}\right)\right)$$

rounds with high probability

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- Match if $\tau = \Theta(T/n)$
  - $O(n^2 \log n)$ rounds
  - If small enough constant, loads within multiplicative factor
- Easy, simple algorithm (Max-Neighbor)

Results: Lower Bound

Theorem: No randomized algorithm can achieve $O(T/n)$-convergence in $o(n^2)$ rounds against an online adaptive adversary.

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- Adversary in each round r:
  - Sees current work distribution $\{x_u(r-1)\}_{u \in V}$
  - Chooses graph $G_r = (V, E_r)$
  - Does not see random coins used by algorithm in round r
- Max-Neighbor upper bound holds

Max Neighbor (round r)

- Node u flips fair coin to decide whether to *send* or receive
  - If send, then u sends proposal to $\text{argmax}_{v \in N(u)}(|x_v(r\text{-}1) - x_u(r\text{-}1)|)$
  - If receive, accept proposal from $\text{argmax}_{v \in S}(|x_v(r\text{-}1) - x_u(r\text{-}1)|)$ (where S is neighbors of u who sent a proposal to u)

- If u accepts proposal from v, they are *connected* in round r
  - Set $x_u(r) = x_v(r) = ½(x_u(r\text{-}1) + x_v(r\text{-}1))$

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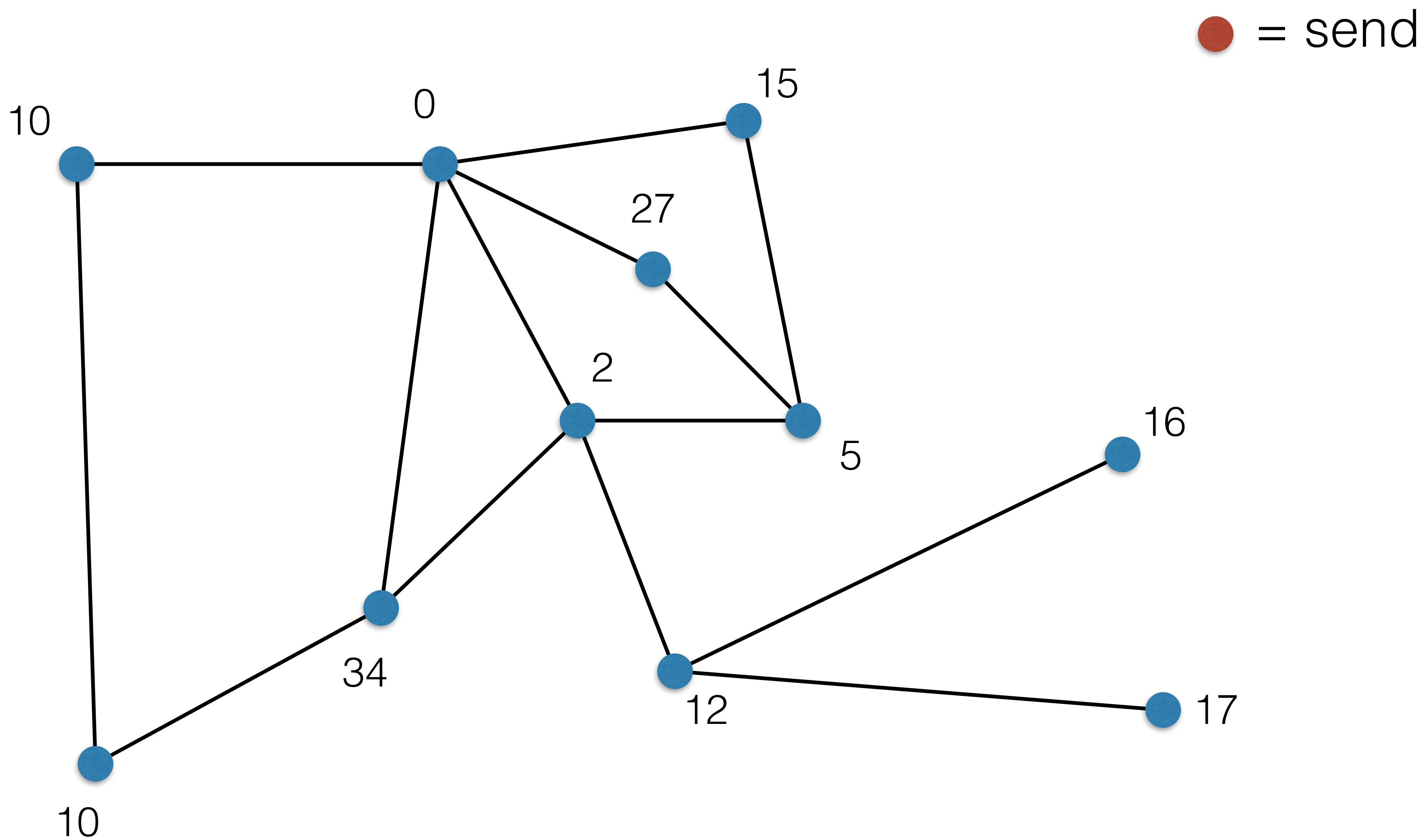

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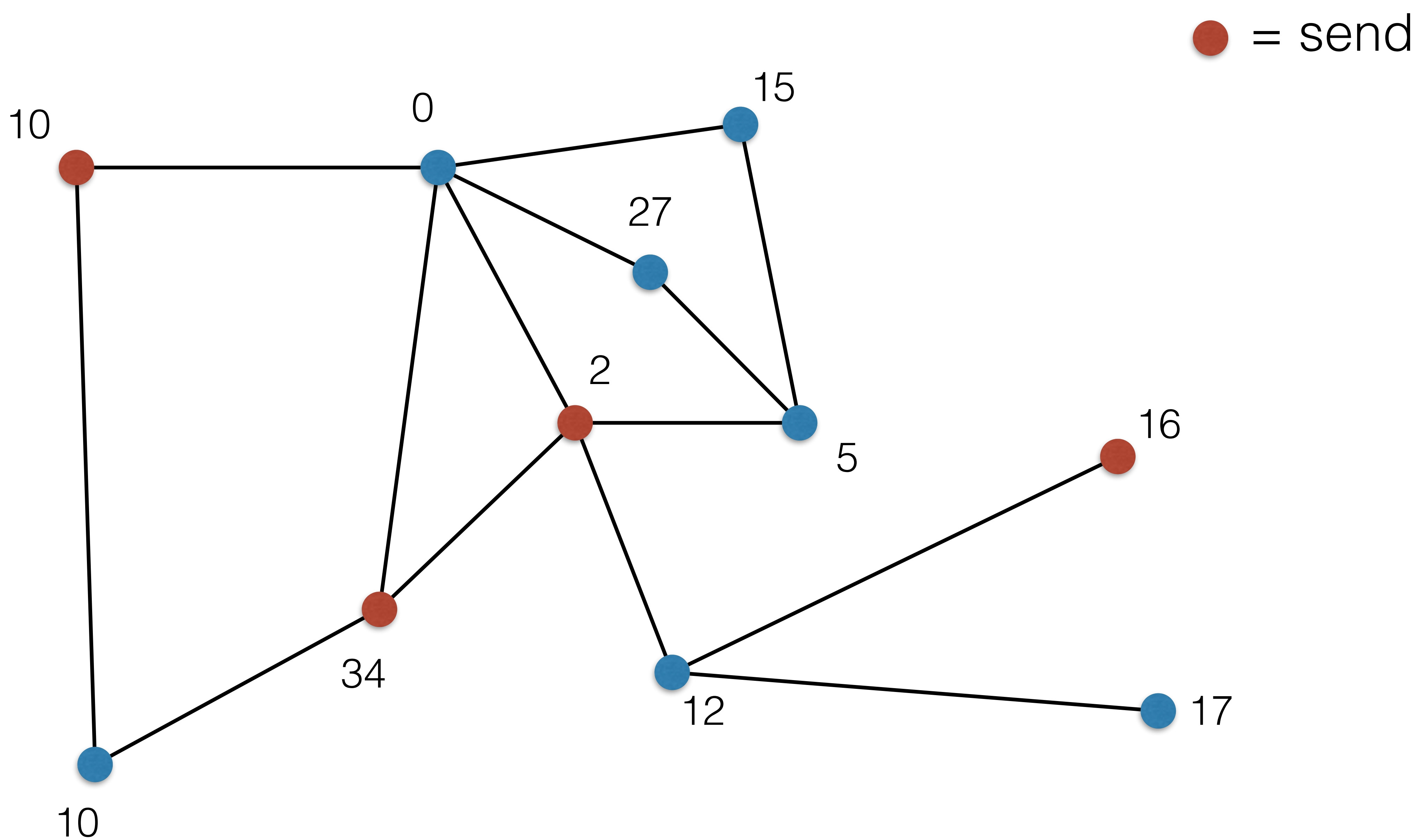

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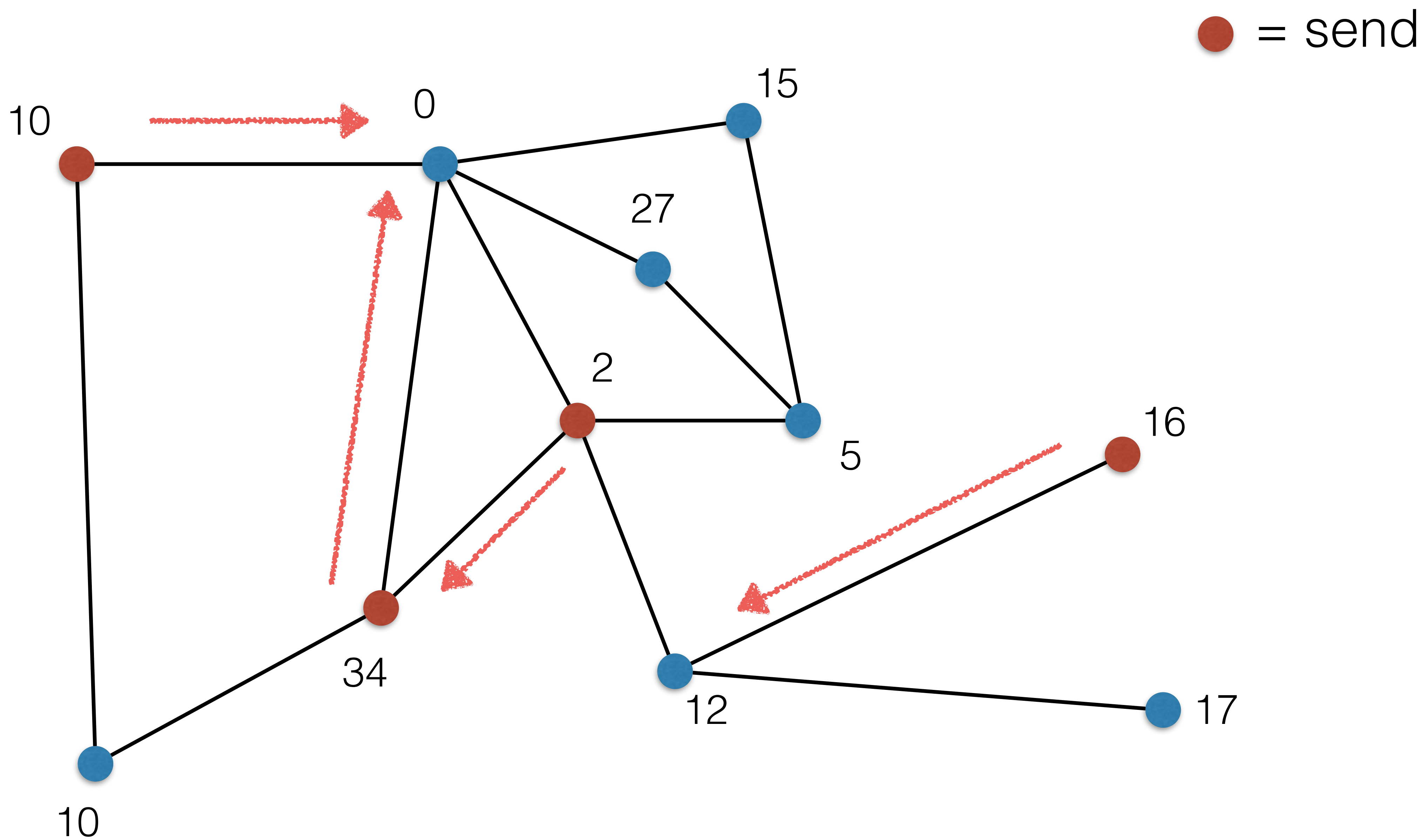

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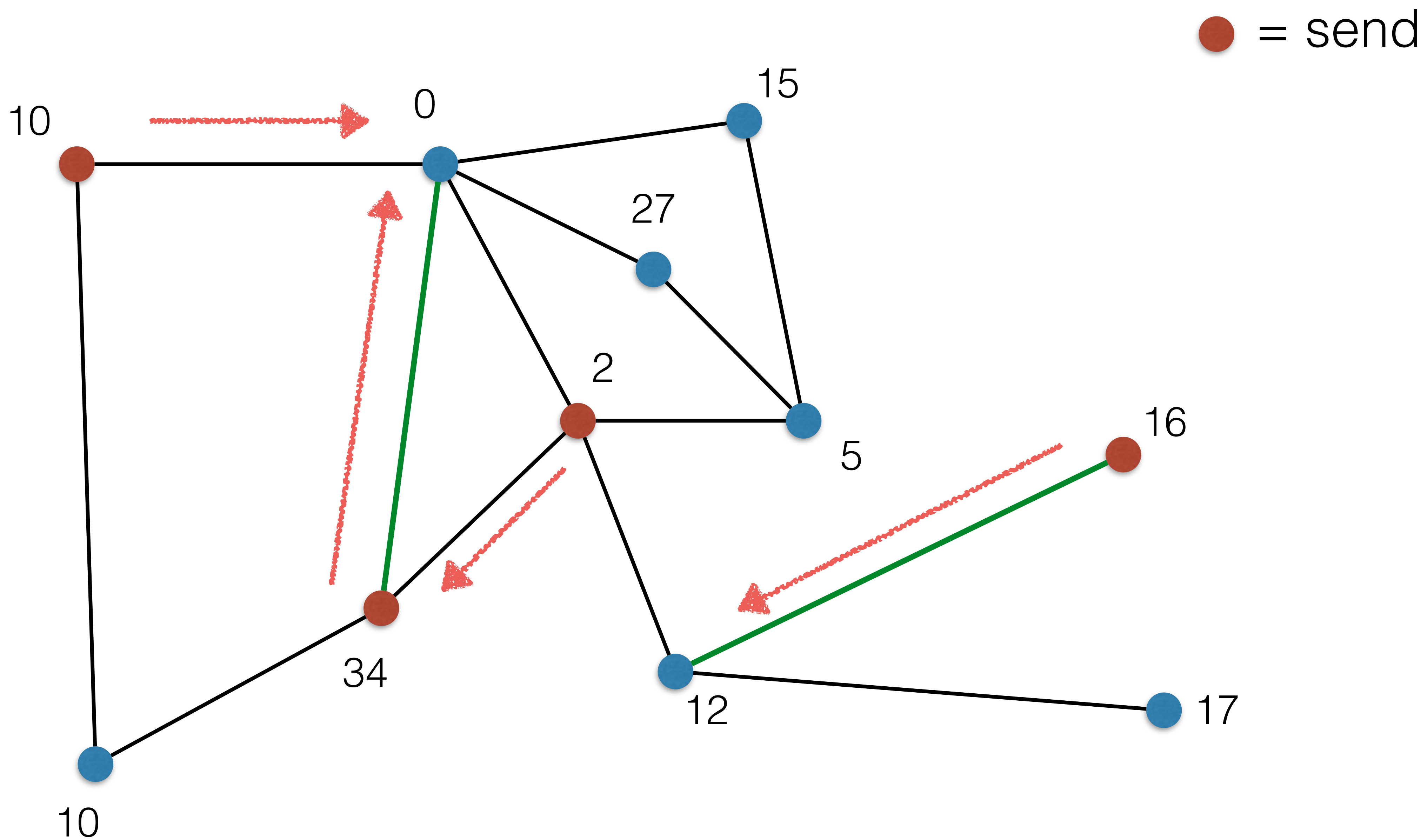

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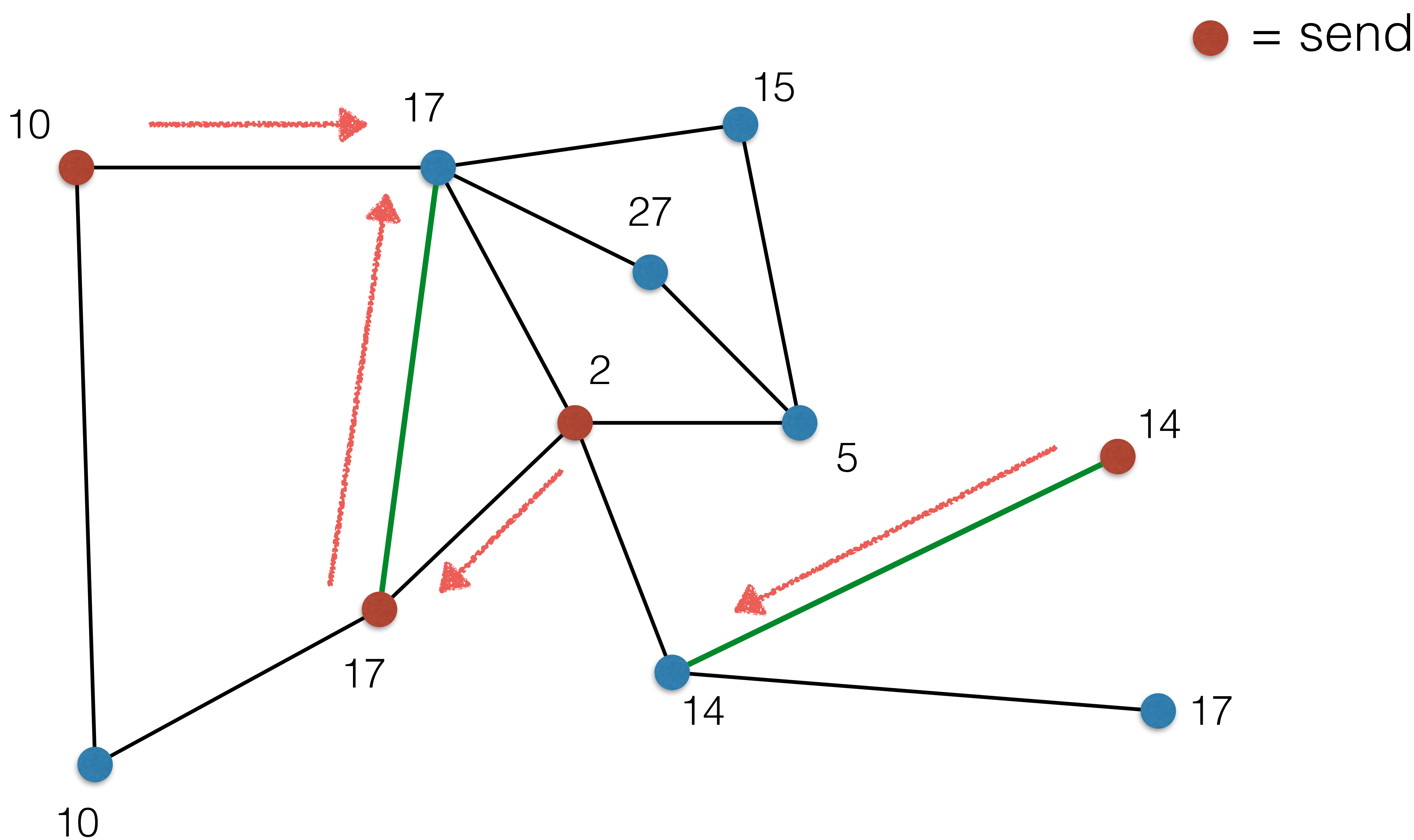

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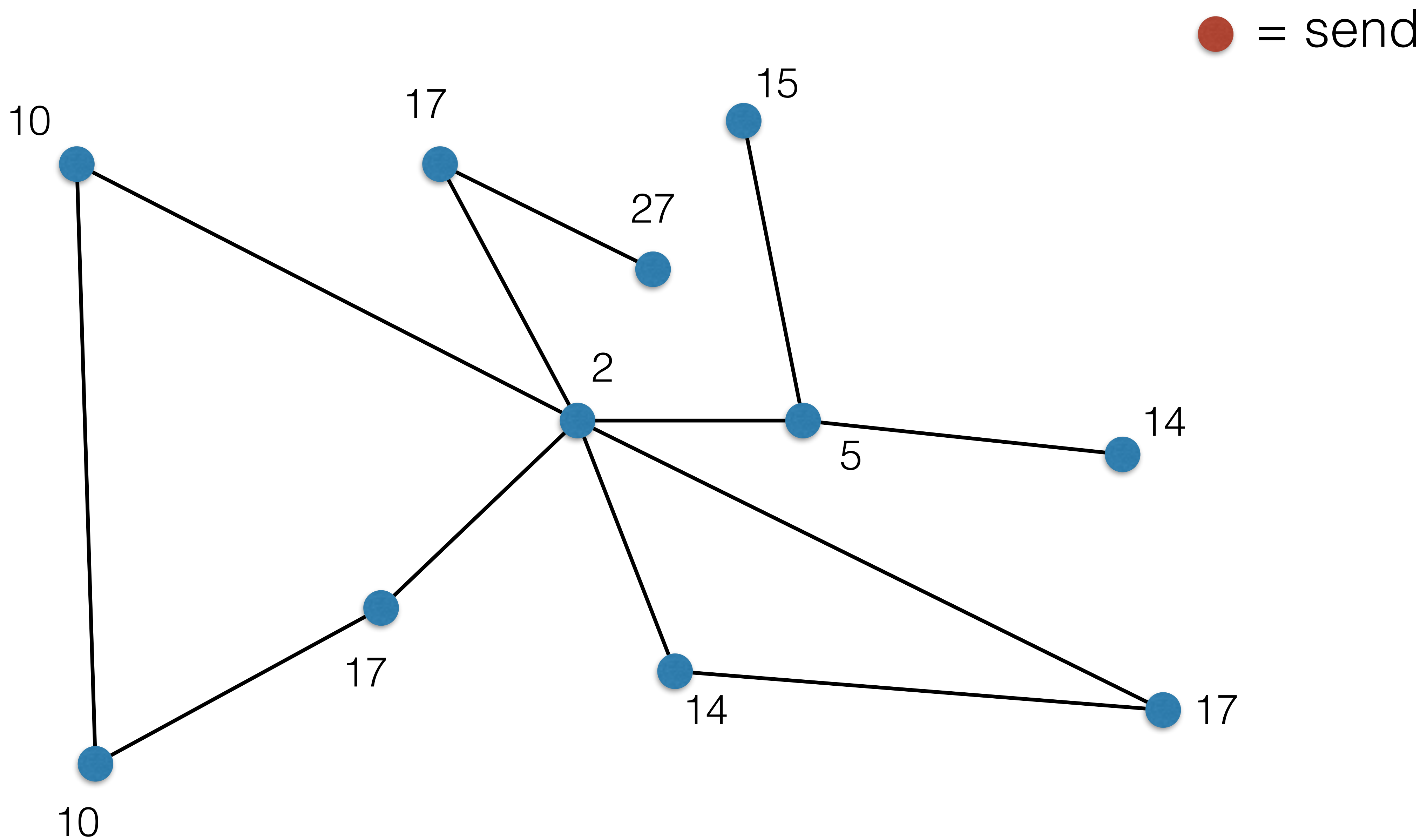

Analysis Outline

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- Potential function $\varphi(r) = \sum_{u,v \in V} \left| x_u(r) - x_v(r) \right|$
  - Initially: $\varphi(0) \leq Tn^2$
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- Want to show potential drops "quickly"
  - Step 1: lower bound potential drop by other function D_r
  - Step 2: with constant probability D_r at least "maximum gap" (good round)
  - Step 3: with high probability, after $O\left(n^2 \log\left(\frac{Tn}{\tau}\right)\right)$ rounds, enough good rounds to drop potential below τ

Step I

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- Lemma: $\varphi(r\text{-}1) - \varphi(r) \geq D_r$
  - $d_{u,v}(r) = |x_u(r) - x_v(r)|$
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  - $D_r = \sum_{\{u,v\}\in Mr} d_{u,v}(r\text{-}1)$

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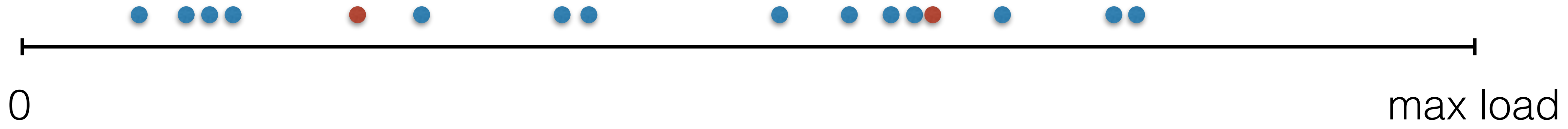

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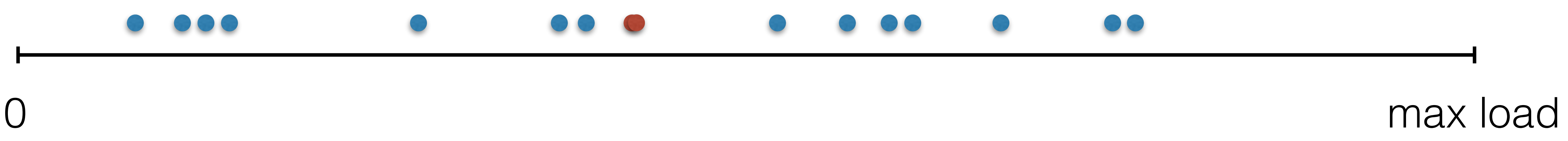

Bounding D_r

- Lemma: $D_r \geq t_{max}(r\text{-}1) / O(\log n)$ with constant probability
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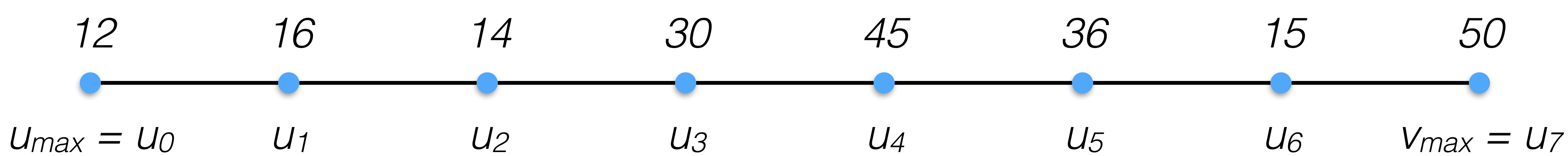

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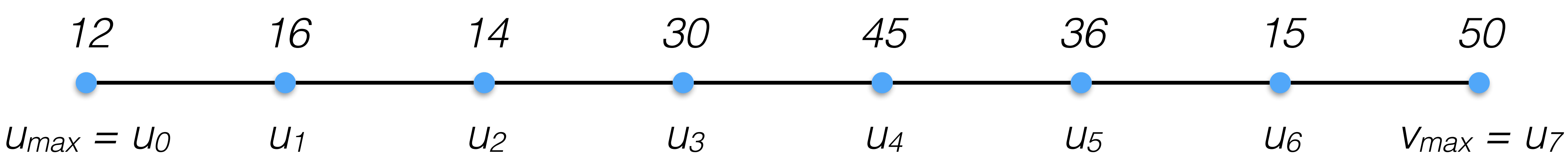

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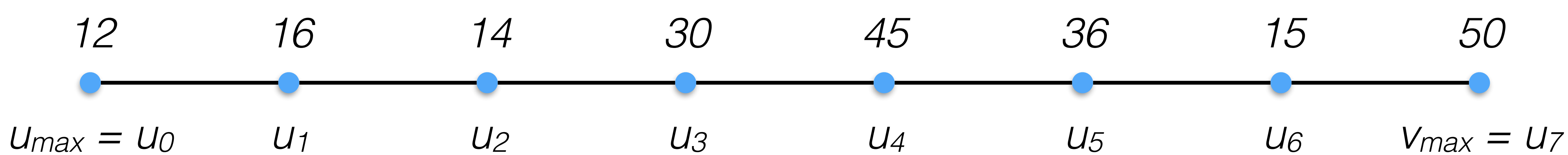

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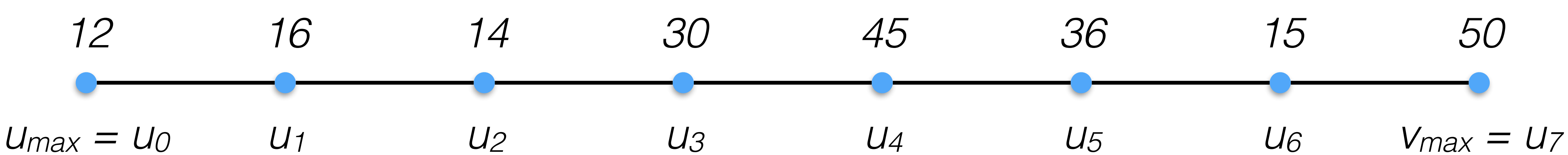
- Would be great if each edge was in matching independently with constant probability, but not true

Bounding D_r

- Fix edge $\{u_i, u_{i+1}\}$. With constant probability there is edge $\{v,w\} \in M_r$ s.t.
  - v, w at distance at most 3 from u,v, and
  - $d_{v,w}(r-1) \geq d_{u_i,u_{i+1}}(r-1)$



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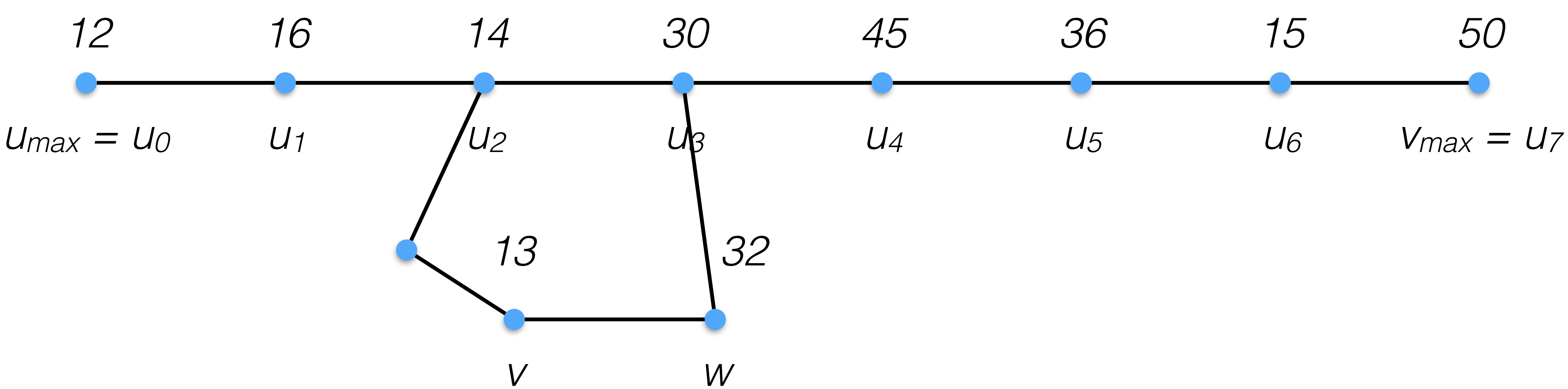

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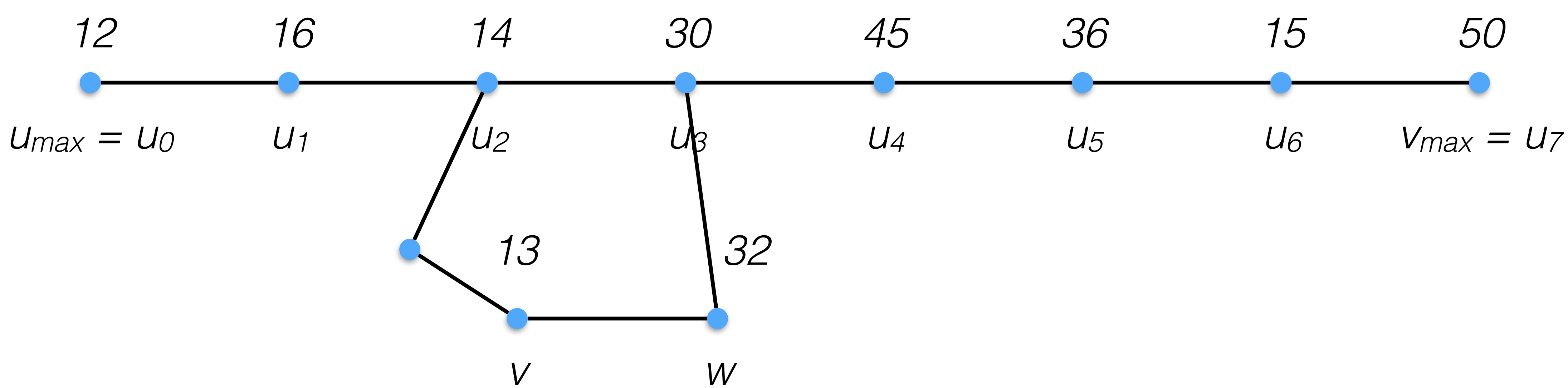

- Independent for i, i' with $|i' - i| > 6$ (distance bound)

- Constant prob. of logarithmic fraction of full path

Putting it Together

- Potential drop at least D_r
- With constant probability, $D_r \geq t_{max}(r\text{-}1) \,/\, O(\log n)$
- Potential at round r at most $n^2\, t_{max}(r\text{-}1)$
- So after about n^2 rounds, potential small enough to guarantee τ-convergence

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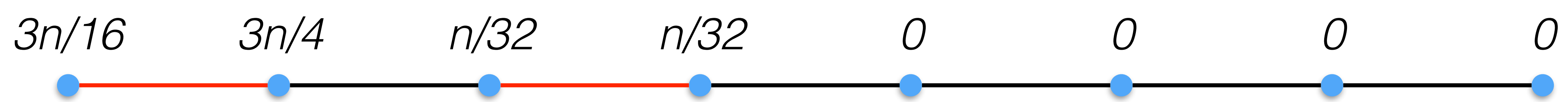

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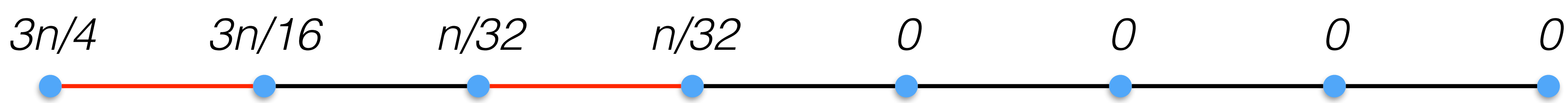

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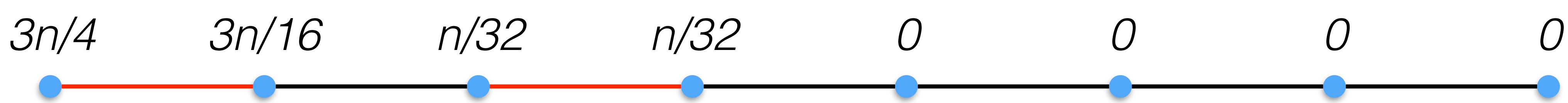

- If ALG used $\{i, i+1\}$ in round r-1, swap (if necessary) so one with smaller work is on the right
  - Lemma: best strategy for ALG is to split work equally across each edge it uses (EQUAL)
  - EQUAL takes $\Omega(n^2)$ rounds before significant weight on node n

Conclusion

- Load balancing upper and lower bounds:
  - Local (no global coordination)
  - At most one connections / node / round (matching)
  - Dynamic networks
  - Provably converges quickly (w.h.p.)
- Lots of interesting questions left!
  - Theory of dynamic graphs
  - Connection to smoothed analysis
  - Logarithmic gap between upper and lower bounds
  - Practice…

Thanks!