Priority Queues

What is a Priority Queue?

Stores prioritized elements

- No notion of storing at particular position

Returns elements in priority order

- Order determined by *key*

What's so different?

Stacks and Queues

- Removal order determined by order of inserting

Sequences

- User chooses exact placement when inserting and explicitly chooses removal order

Priority Queue

- Order determined by key
- Key may be part of element data or separate

What's it good for?

Order of returned elements is not FIFO or LIFO (as in queue or stack)

Random access not necessary (as in sequence) or desirable

Examples

- Plane landings managed by air traffic control
- Processes scheduled by CPU
- College admissions process for students
  - —What are some of the criteria?

College Admissions Key

Student submits:

- Personal data (geography, is parent alum?, activities?)
- Transcript
- Essays
- Standardized test scores
- Recommendations

Admissions agent:

- Each datum converted to number
- Formula converts to single numeric key

Student selection process

Simple scheme

- Collect applications until due date
- Sort by keys
- Take top *k* students

More realistic

- Prioritize applications as they come in
- Accept some top students ASAP
- Maybe even change data/key as you go

Priority Queue ADT

insertItem(k,e): insert element e with key k

extractMin( ): return element with minimum key and remove from queue

minElement( ): return (look at) min element

minKey( ): return minimum key

size( ): return number of elements

isEmpty( ): size == 0?

Keys, Comparitors and Total Orders

Key type needs comparison operator (returns boolean) with following properties:

- Reflexive: $k \le k$
- Antisymmetric: $(k1 \le k2)$ && $(k2 \le k1) \rightarrow k1 = k2$
- Transitive: $((k1 \le k2)$ && $(k2 \le k3) \rightarrow k1 \le k3$

These properties guarantee consistent, *total ordering*

Abstracting Comparitors

Allows for different types of comparison

- e.g. Numeric vs. lexicographic (for strings)

Several approaches possible

- Build PQ object to know about specific key type and comparison
- Build key object to know about comparison
- Build separate comparitor object for each type of comparison

Book argues for #3, but I also recommend #2

Implementing PQ with Unsorted Sequence

Each call to insertItem(k, e) uses insertLast( ) to store in Sequence

- $O(1)$ time

Each call to extractMin( ) traverses the entire sequence to find the minimum, then removes element

- $O(n)$ time

Implementing PQ with Sorted Sequence

Each call to insertItem(k, e) traverses sorted sequence to find correct position, then does insert

- O(n) worst case

Each call to extractMin( ) does removeFirst( )

- $O(1)$ time

Sorting Using a PQ

Elements begin in arbitrary order in a sequence

Move elements from sequence into PQ

Extract elements from PQ and reinsert into sequence in priority order

Analysis depends on implementation choices

Analyzing Queue Efficiency for Sorting

N insertElement( ) operations followed by N extractMin( ) operations

Selection Sort

PQ sorting using unsorted sequence

Insert all *n* items in input order

Extract by *selecting* min item *n* times

Insertion Sort

PQ sorting using sorted sequence

Sequentially *insert* items into sequence in sorted order

Extract items easily from sorted sequence

Sort Analysis

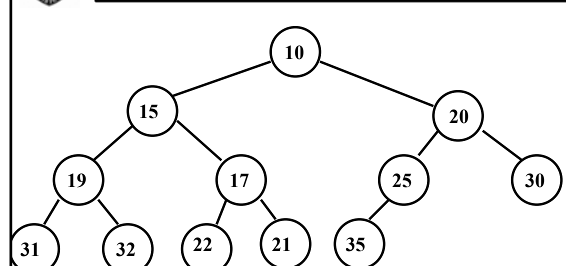
		Sel. Sort	Ins. Sort
`foreach element, E_i, in S`	$O(n)$		
`PQ.insert(E_i)`		$\sum_{i=0}^{i=n-1} O(1)$	$\sum_{i=0}^{i=n-1} O(i)$
`while !PQ.empty()`	$O(n)$		
`PQ.extractMin()`		$\sum_{i=0}^{i=n-1} O(i)$	$\sum_{i=0}^{i=n-1} O(1)$

$$O(n) + \sum_{i=0}^{i=n-1} O(1) + \sum_{i=0}^{i=n-1} O(i) = O(n) + O(n) + O(n^2) = O(n^2)$$

Heap

- Binary tree-based data structure
  - *Complete* in the sense that it fills up levels as completely as possible
  - Height of tree is *O*(logn)
- Stores elements with keys
- All nodes satisfy the *heap property*:
  - The key value at a node is less than or equal to the key value of the node's children
- Allows insertItem( ) and extractMin( ) in *O*(logn) time

Heap Example

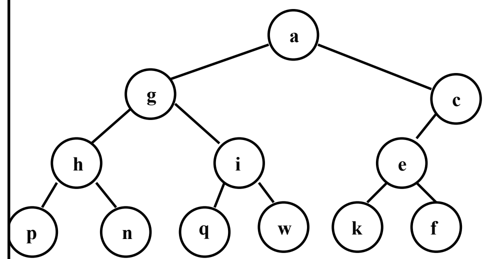


PQ Quiz Show!

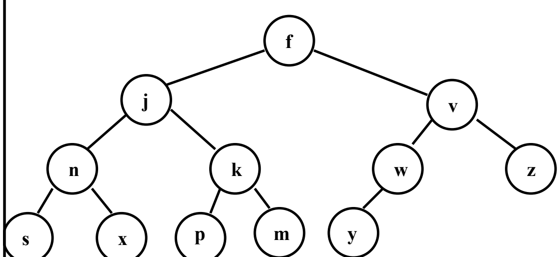
Heap, or Not A Heap?

(no paper necessary)

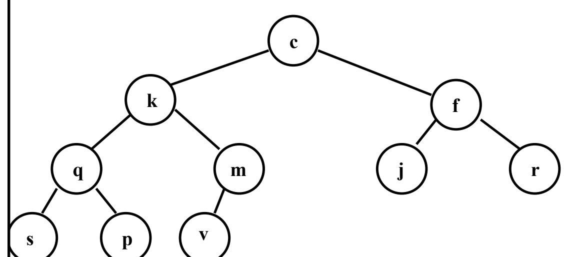
Heap, or Not a Heap?



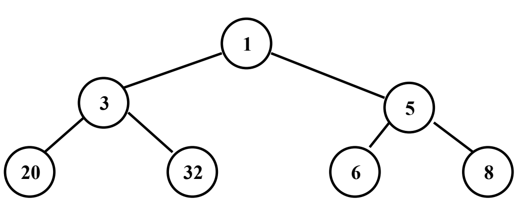
Heap, or Not a Heap?



Heap, or Not a Heap?



Heap, or Not a Heap?



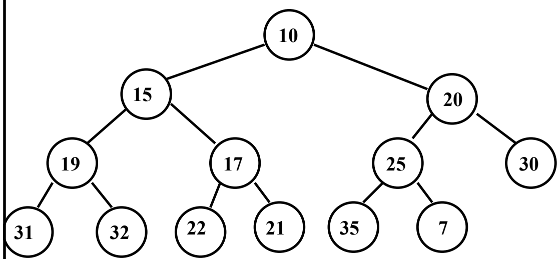
Inserting into Heap

- Create new node as “last” element
- Insert key/element into new node
- Bubble node upward until heap property is satisfied

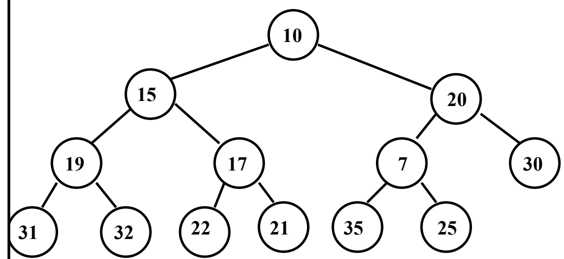
```
while (!isRoot(node) &&
  (node.key < node.parent.key))
     swap(node, parent)
```

(just pseudocode - can’t do it exactly like this in Java)

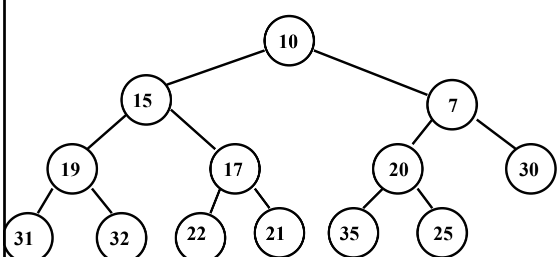
Heap Insert Example



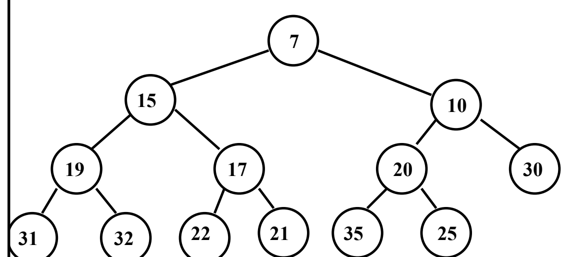
Bubble Upward



Bubble Upward



Bubble Upward



Heap Insert Analysis

New node always inserted at lowest level

Node bubbles upward

- **up to root in worst case**
- **path length to root is *O*(log*n*)**

Total time for insert is *O*(log*n*)

Extracting from Heap

Copy element from root node

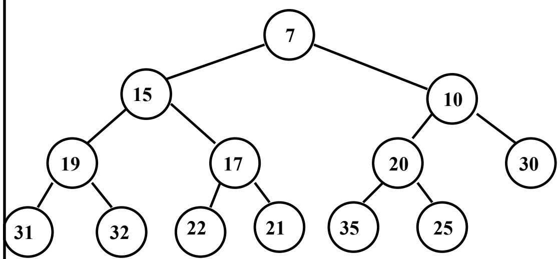
Copy element/key from last node to root node

Delete last node

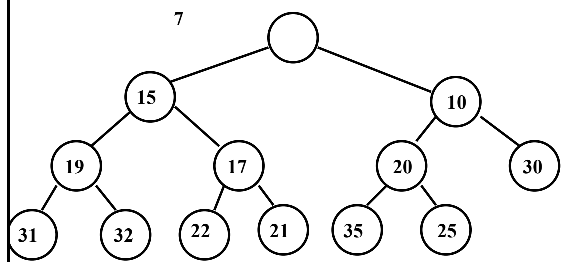
Bubble root node downward until heap property satisfied

```
while (!isExternal(node) &&
        (node.key > node.smallestChild.key))
    swap(node, node.smallestChild)
```

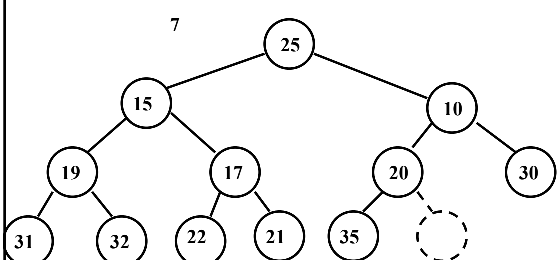
Heap Extract Example



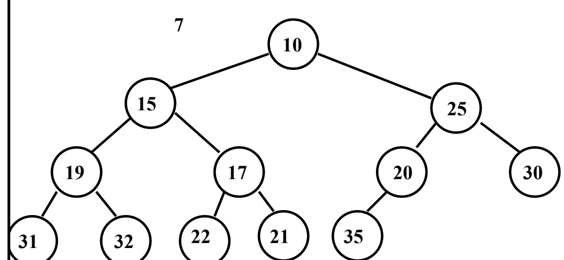
Copy Extracted Element



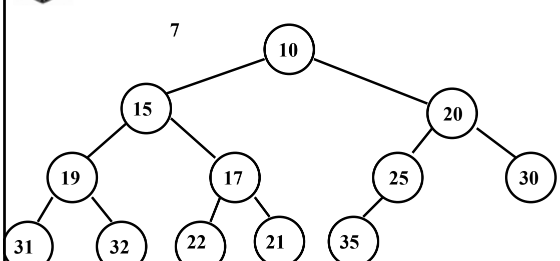
Move last node to root



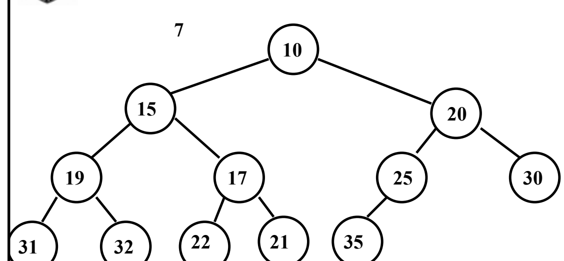
Bubble Downward



Bubble Downward



All done



Heap Extract Analysis

Again, each swap takes constant time

Maximum swaps is path length from root to leaf

- **→ Total work is log*n* * *O*(1) = *O*(log*n*)**

Sort Analysis

Heap Sort

`foreach element, E_i, in S`	$O(n)$
`PQ.insert(E_i)`	$\sum_{i=0}^{i=n-1} O(\log i)$
`while !PQ.empty()`	$O(n)$
`PQ.extractMin()`	$\sum_{i=0}^{i=n-1} O(\log i)$

$$O(n) + 2\sum_{i=0}^{i=n-1} O(\log i) < O(n) + 2\sum_{i=0}^{i=n-1} O(\log n)$$

$$= O(n) + 2n*O(\log n) = O(n\log n)$$

(showing $\theta(n\log n)$ is a bit harder)

In-class Exercise

What does the heap look like after the following sequence of insertions:

5 30 2 15 7 45 20 6 18