# Zero-Knowledge Proofs

CS 601.642/442 Modern Cryptography

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# What is a Proof?

- An argument (or sufficient evidence) that can convince a reader of the truth of some statement
- Mathematical proof: Deductive argument for a statement, by reducing the validity of the statement to a set of axioms or assumptions
- Desirable features in a proof:
  - The verifier should accept the proof if the statement is true
  - The verifier should reject *any* proof if the statement is false
  - Proof must be finite (or succinct) and efficiently verifiable
- E.g., Proof that there are infinitely many primes should not simply be a list of all the primes. Not only would it take forever to generate that proof, it would also take forever to verify it

# What is a Proof? (contd.)

1. Question 1: How to model efficient verifiability?
   - Verifier must be polynomial time in the length of the statement
2. Question 2: Must a proof be *non-interactive*?
   - Or can a proof be a conversation? (i.e., *interactive*)

# Interactive Protocols

- Interactive Turing Machine (ITM): A Turing machine with two additional tapes: a read-only communication tape for receiving messages, a write-only communication tape for sending messages.
- An interactive protocol $(M_1, M_2)$ is a pair of ITMs that share communication tapes s.t. the send-tape of the first ITM is the receive-tape of the second, and vice-versa
- Protocol proceeds in rounds. In each round, only one ITM is active, the other is idle. Protocol ends when both ITMs *halt*
- $M_1(x_1, z_1) \leftrightarrow M_2(x_2, z_2)$: A (randomized) protocol execution where $x_i$ is input and $z_i$ is auxiliary input of $M_i$
- $\mathsf{Out}_{M_i}(e)$: Output of $M_i$ in an execution $e$
- $\mathsf{View}_{M_i}(e)$: View of $M_i$ in an execution $e$ consists of its input, random tape, auxiliary input and all the protocol messages it sees.

# Interactive Proofs

## Definition (Interactive Proofs)

A pair of ITMs $(P, V)$ is an interactive proof system for a language $L$ if $V$ is a PPT machine and the following properties hold:

- **Completeness:** For every $x \in L$,

$$\Pr\left[\mathsf{Out}_V[P(x) \leftrightarrow V(x)] = 1\right] = 1$$

- **Soundness:** There exists a negligible function $\nu(\cdot)$ s.t. $\forall x \notin L$ and for all adversarial provers $P^*$,

$$\Pr\left[\mathsf{Out}_V[P^*(x) \leftrightarrow V(x)] = 1\right] \leqslant \nu(|x|)$$

<u>Remark</u>: In the above definition, prover is not required to be efficient. Later, we will also consider efficient provers.

# Why Interactive proofs?

- Let $L$ be a language in **NP** and let $R$ be the associated relation
- For any $x \in L$, there exists a "small" (polynomial-size) witness $w$
- By checking that $R(x, w) = 1$, we can verify that $x \in L$
- Therefore, $w$ is a *non-interactive* proof for $x$
- E.g. Graph Isomorphism: Two graphs $G_0$ and $G_1$ are isomorphic if there exists a permutation $\pi$ that maps the vertices of $G_0$ onto the vertices of $G_1$.

So why use interactive proofs after all?

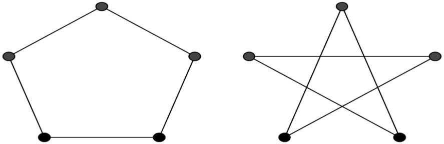
# Why Interactive proofs? (contd.)

Two main reasons for interaction:

1. Proving statements in languages not known to be in **NP**
   - Single prover [Shamir]: $\mathbf{IP} = \mathbf{PSPACE}$
   - Multiple provers [Babai-Fortnow-Lund]: $\mathbf{MIP} = \mathbf{NEXP}$
2. Achieving privacy guarantee for prover
   - Zero knowledge [Goldwasser-Micali-Rackoff]: Verifier learns nothing from the proof beyond the validity of the statement!

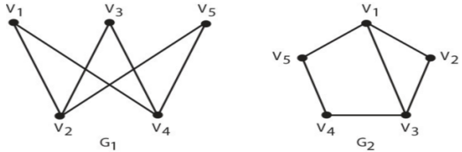
# Notation for Graphs

- Graph $G = (V, E)$ where $V$ is set of vertices and $E$ is set of edges
- $|V| = n$, $|E| = m$
- $\Pi_n$ is the set of all permutations $\pi$ over $n$ vertices
- Graph Isomorphism: $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ are isomorphic if there exists a permutation $\pi$ s.t.:
  - $V_1 = \{\pi(v) \mid v \in V_0\}$
  - $E_1 = \{(\pi(v_1), \pi(v_2)) \mid (v_1, v_2) \in E_0\}$
  - Alternatively, $G_1 = \pi(G_0)$
  - Graph Isomorphism is in **NP**



# Notation for Graphs (contd.)

- Graph Non-Isomorphism: $G_0$ and $G_1$ are non-isomorphic if there exists no permutation $\pi \in \Pi_n$ s.t. $G_1 = \pi(G_0)$




- Graph Non-Isomorphism is in **co-NP**, and not known to be in **NP**

# How to Prove Graph Non-Isomorphism?

- Suppose $P$ wants to prove to $V$ that $G_0$ and $G_1$ are not isomorphic
- One way to prove this is to write down all possible permutations $\pi$ over $n$ vertices and show that for every $\pi$, $G_1 \neq \pi(G_0)$. However, this is not efficiently verifiable
- How to design an efficiently verifiable interactive proof?

# Interactive Proof for Graph Non-Isomorphism

**Common Input:** $x = (G_0, G_1)$

**Protocol** $(P, V)$**:** Repeat the following procedure $n$ times using fresh randomness

$V \rightarrow P$: $V$ chooses a random bit $b \in \{0, 1\}$ and a random permutation $\pi \in \Pi_n$. It computes $H = \pi(G_b)$ and sends $H$ to $P$

$P \rightarrow V$: $P$ computes $b'$ s.t. $H$ and $G_{b'}$ are isomorphic and sends $b'$ to $V$

$V(x, b, b')$: $V$ outputs 1 if $b' = b$ and 0 otherwise

# $(P, V)$ is an Interactive Proof

- **Completeness:** If $G_0$ and $G_1$ are not isomorphic, then an unbounded prover can always find $b'$ s.t. $b' = b$
- **Soundness:** If $G_0$ and $G_1$ are isomorphic, then $H$ is isomorphic to both $G_0$ and $G_1$! Therefore, in one iteration, any (unbounded) prover can correctly guess $b$ with probability at most $\frac{1}{2}$. Since each iteration is independent, prover can succeed in all iterations with probability at most $2^{-n}$.

# Interactive Proofs with Efficient Provers

- Prover in graph non-isomorphism protocol is inefficient
- Want: Interactive Proofs with *efficient* provers
- We will restrict attention to languages in **NP**
- Prover strategy must be efficient when it is given a witness $w$ for a statement $x$ that it attempts to prove

**Definition**

An interactive proof system $(P, V)$ for a language $L$ with witness relation $R$ is said to have an *efficient prover* if $P$ is PPT and the completeness condition holds for every $w \in R(x)$

Remark: Even though honest $P$ is efficient, we still require soundness guarantee against *all* adversarial provers

# Interactive Proof for Graph Isomorphism

- Recall: to prove that $G_0$ and $G_1$ are isomorphic, $P$ can simply send $\pi$ s.t. $G_1 = \pi(G_0)$
- If $P$ is given $\pi$ as input, then it is also efficient
- However, in this protocol, $V$ learns the permutation $\pi$. Now, it can also prove to someone else that $G_0$ and $G_1$ are isomorphic
- Can we construct an interactive proof that hides the witness $\pi$ from $V$?
- Or better yet, can we construct an interactive proof that that only reveals the validity of the statement to $V$ and *nothing else*?
- Sounds paradoxical, right?
- Goldwasser, Micali, Rackoff showed that it can be done!

# Interactive Proof for Graph Isomorphism

**Common Input:** $x = (G_0, G_1)$

$P$**'s witness:** $\pi$ **s.t.** $G_1 = \pi(G_0)$

**Protocol** $(P, V)$**:** Repeat the following procedure $n$ times using fresh randomness

$P \rightarrow V$: Prover chooses a random permutation $\sigma \in \Pi_n$, computes $H = \sigma(G_0)$ and sends $H$

$V \rightarrow P$: $V$ chooses a random bit $b \in \{0, 1\}$ and sends it to $P$

$P \rightarrow V$: If $b = 0$, $P$ sends $\sigma$. Otherwise, it sends $\varphi = \sigma \cdot \pi^{-1}$

$V(x, b, \varphi)$: $V$ outputs 1 iff $H = \varphi(G_b)$

# $(P, V)$ is an Interactive Proof

- **Completeness:** If $G_0$ and $G_1$ are isomorphic, then $V$ always accepts since $\sigma(G_0) = H$ and $\sigma(\pi^{-1}(G_1)) = \sigma(G_0) = H$
- **Soundness:** If $G_0$ and $G_1$ are *not* isomorphic, then $H$ is isomorphic to either $G_0$ or $G_1$, but not both! Since $b$ is chosen at random after $H$ is fixed, with probability $\frac{1}{2}$, $H$ is not isomorphic to $G_b$. Thus, an adversarial prover can succeed with probability at most $\frac{1}{2}$. Since each iteration is independent, prover can succeed in all iterations with probability at most $2^{-n}$.

# Towards Zero Knowledge

- The graph isomorphism protocol also has the property that $V$ does not gain any knowledge from its interaction with $P$ beyond the fact that $G_0$ and $G_1$ are isomorphic
- In particular, $V$'s witness $\pi$ remains private from $P$
- Q. 1: How to formalize "does not gain any knowledge?"
- Q. 2: What is knowledge?

# Towards Zero Knowledge (contd.)

Rules for formalizing "(zero) knowledge":

Rule 1: Randomness is for free

Rule 2: Polynomial-time computation is for free

That is, by learning the result of a random process or result of a polynomial time computation, we gain no knowledge

# When is knowledge conveyed?

Scenario 1: Someone tells you he will sell you a 100-bit random string for \$1000.

Scenario 2: Someone tells you he will sell you the product of two prime numbers of your choice for \$1000.

Scenario 3: Someone tells you he will sell you the output of an exponential time computation (e.g., isomorphism between two graphs) for \$1000.

Think: Should you accept any of these offers?

We can generate 100-bit random string for free by flipping a coin, and we can also multiply on our own for free. But an exponential-time computation is hard to perform on our own, since we are PPT. So we should reject first and second offers, but seriously consider the third one!

# Zero Knowledge: Intuition

- We do not gain any knowledge from an interaction if we could have carried it out on our own

- Intuition for ZK: $V$ can generate a protocol transcript on its own, without talking to $P$. If this transcript is indistinguishable from a real execution, then clearly $V$ does not learn anything by talking to $P$

- Formalized via notion of *Simulator*, as in definition of semantic security for encryption

# Zero Knowledge: Definition I

**Definition (Honest Verifier Zero Knowledge)**

An interactive proof $(P, V)$ for a language $L$ with witness relation $R$ is said to be *honest verifier zero knowledge* if there exists a PPT simulator $S$ s.t. for every non-uniform PPT distinguisher $D$, there exists a negligible function $\nu(\cdot)$ s.t. for every $x \in L$, $w \in R(x)$, $z \in \{0,1\}^*$, $D$ distinguishes between the following distributions with probability at most $\nu(n)$:

- $\left\{\mathsf{View}_V[P(x,w) \leftrightarrow V(x,z)]\right\}$
- $\left\{S(1^n, x, z)\right\}$

# Remarks on the Definition

- Captures that whatever $V$ "saw" in the interactive proof, it could have generated it on its own by running the simulator $S$
- The auxiliary input to $V$ captures any a priori information $V$ may have about $x$. Definition promises that $V$ does not learn anything "new"
- Problem: However, the above is promised only if verifier $V$ follows the protocol
- What if $V$ is malicious and deviates from the honest strategy?
- Want: Existence of a simulator $S$ for every, possibly malicious (efficient) verifier strategy $V^*$
- For now, will relax the simulator and allow it to be *expected* PPT, i.e., a machine whose expected running time is polynomial

# Zero Knowledge: Definition II

**Definition (Zero Knowledge)**

An interactive proof $(P, V)$ for a language $L$ with witness relation $R$ is said to be *zero knowledge* if for every non-uniform PPT adversary $V^*$, there exists an expected PPT simulator $S$ s.t. for every non-uniform PPT distinguisher $D$, there exists a negligible function $\nu(\cdot)$ s.t. for every $x \in L$, $w \in R(x)$, $z \in \{0,1\}^*$, $D$ distinguishes between the following distributions with probability at most $\nu(n)$:

- $\left\{ \mathsf{View}^*_V[P(x,w) \leftrightarrow V^*(x,z)] \right\}$
- $\left\{ S(1^n, x, z) \right\}$

- If the distributions are statistically close, then we call it *statistical zero knowledge*
- If the distributions are identical, then we call it *perfect zero knowledge*