Non-Interactive Zero Knowledge (I)

CS 600.442 Modern Cryptography

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The Setting

- Alice wants to prove an **NP** statement to Bob without revealing her private witness
- However, Alice only has the resource to send a *single* message to Bob. Therefore, they cannot run an interactive zero-knowledge proof
- To make matters worse, 1-message zero-knowledge is only possible for languages in **BPP**! (Think: Why?)
- Fortunately, they both have access to a *common random string* that was (honestly) generated by someone they both trust
- Can Alice prove statements *non-interactively* to Bob using the common random string?

Non-Interactive Proofs

Syntax. A non-interactive proof system for a language L with witness relation R is a tuple of algorithms $(\mathsf{K}, \mathsf{P}, \mathsf{V})$ such that:

- **Setup:** $\sigma \leftarrow \mathsf{K}(1^n)$ outputs a common random string
- **Prove:** $\pi \leftarrow \mathsf{P}(\sigma, x, w)$ takes as input a common random string σ, a statement $x \in L$ and a witness w and outputs a proof π
- **Verify:** $V(\sigma, x, \pi)$ outputs 1 if it accepts the proof and 0 otherwise

A non-interactive proof system must satisfy completeness and soundness properties

Non-Interactive Proofs (contd.)

Completeness: $\forall x \in L$, $\forall w \in R(x)$:

$$\Pr\left[\sigma \leftarrow \mathsf{K}(1^n); \pi \leftarrow \mathsf{P}(\sigma, x, w) : \mathsf{V}(\sigma, x, \pi) = 1\right] = 1$$

Non-Adaptive Soundness: There exists a negligible function $\nu(\cdot)$ s.t. $\forall x \notin L$:

$$\Pr\left[\sigma \leftarrow \mathsf{K}(1^n); \exists\ \pi \text{ s.t. } \mathsf{V}(\sigma, x, \pi) = 1\right] \leqslant \nu(n)$$

Adaptive Soundness: There exists a negligible function $\nu(\cdot)$ s.t.:

$$\Pr\left[\sigma \leftarrow \mathsf{K}(1^n); \exists\ (x, \pi) \text{ s.t. } x \notin L \wedge \mathsf{V}(\sigma, x, \pi) = 1\right] \leqslant \nu(n)$$

Note: In non-adaptive soundness, the adversary chooses x before seeing the common random string whereas in adaptive soundness, it can choose x depending upon the common random string

Non-Interactive Zero Knowledge (NIZK)

Definition (Non-Adaptive NIZK)

A non-interactive proof system $(\mathsf{K}, \mathsf{P}, \mathsf{V})$ for a language L with witness relation R is *non-adaptive zero-knowledge* if there exists a PPT simulator $\mathcal{S}$ s.t. for every $x \in L$, $w \in R(x)$, the output distributions of the following two experiments are computationally indistinguishable:

$\mathsf{REAL}(1^n, x, w)$	$\mathsf{IDEAL}(1^n, x)$
$\sigma \leftarrow \mathsf{K}(1^n)$	$(\sigma, \pi) \leftarrow \mathcal{S}(1^n, x)$
$\pi \leftarrow \mathsf{P}(\sigma, x, w)$	
Output (σ, π)	Output (σ, π)

Note: The simulator generates both the common random string and the simulated proof given the statement x is input. In particular, the simulated common random string can depend on x and can therefore only be used for a single proof

Non-Interactive Zero Knowledge (contd.)

Definition (Adaptive NIZK)

A non-interactive proof system $(\mathsf{K}, \mathsf{P}, \mathsf{V})$ for a language L with witness relation R is *adaptive zero-knowledge* if there exists a PPT simulator $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1)$ s.t. for every $x \in L$, $w \in R(x)$, the output distributions of the following two experiments are computationally indistinguishable:

$\mathsf{REAL}(1^n, x, w)$	$\mathsf{IDEAL}(1^n, x)$
$\sigma \leftarrow \mathsf{K}(1^n)$	$(\sigma, \tau) \leftarrow \mathcal{S}_0(1^n)$
$\pi \leftarrow \mathsf{P}(\sigma, x, w)$	$\pi \leftarrow \mathcal{S}_1(\sigma, \tau, x)$
Output (σ, π)	Output (σ, π)

Note 1: Here, τ is a "trapdoor" for the simulated common random string σ that is used by $\mathcal{S}_1$ to generate an accepting proof for x without knowing the witness.

Note 2: This definition captures *reusable* common random strings

Remarks on NIZK Definition

- In NIZK, the simulator is given “extra power” to choose the common random string, along with possibly a trapdoor to enable simulation without a witness
- In interactive ZK, the extra power to the simulator was the ability to “reset” the verifier
- Indeed, a simulator must always have some extra power over the normal prover, otherwise, the definition would be impossible to realize for languages outside **BPP**
- In NIZKs, the extra power is ok since we require indistinguishability of the “joint distribution” over the common random string and the proof

From Non-Adaptive to Adaptive Soundness

Lemma

There exists an efficient transformation from any non-interactive proof system $(\mathsf{K}, \mathsf{P}, \mathsf{V})$ *with non-adaptive soundness into a non-interactive proof system* $(\mathsf{K}', \mathsf{P}', \mathsf{V}')$ *with adaptive soundness*

Proof Strategy: Let $\ell(n)$ be the length of the statements

- Repeat $(\mathsf{K}, \mathsf{P}, \mathsf{V})$ polynomially many times (with fresh randomness) so that soundness error decreases to $2^{-2\ell(n)}$
- Non-adaptive soundness means that a randomly sampled σ is "bad" for a statement x with probability $2^{-2\ell(n)}$
- By Union Bound, σ is "bad" for all statements with probability $2^{-\ell(n)}$. Therefore, we have adaptive soundness

NIZKs for **NP**

I. Non-adaptive Zero Knowledge: We first construct NIZKs for **NP** with non-adaptive zero-knowledge property using the following two steps:

Step 1. Construct a NIZK proof system for **NP** in the **hidden-bit model**. This step is unconditional

Step 2. Using trapdoor permutations, transform any NIZK proof system for language in the hidden-bit model to a non-adaptive NIZK proof system in the common random string model

II. Adaptive Zero Knowledge: Next, we transform non-adaptive NIZKs for **NP** into adaptive NIZKs for **NP**. This step only requires one-way functions, which are implied by trapdoor permutations.

Putting all the steps together, we obtain adaptive NIZKs for **NP** based on trapdoor permutations

Roadmap

- **Today:** Defining NIZKs in hidden-bit model, and transformation from NIZKs in hidden-bit model to NIZKs in common random string model
- **Next time:** NIZKs for **NP** in the hidden-bit model
- **Homework:** Non-adaptive NIZKs to Adaptive NIZKs

NIZK in Hidden-Bit Model

Syntax. A non-interactive proof system for a language L with witness relation R in the hidden-bit model is a tuple of algorithms $(\mathsf{K}_{\mathsf{HB}}, \mathsf{P}_{\mathsf{HB}}, \mathsf{V}_{\mathsf{HB}})$ such that:

- **Setup:** $r \leftarrow \mathsf{K}_{\mathsf{HB}}(1^n)$ outputs the hidden random string
- **Prove:** $(I, \pi) \leftarrow \mathsf{P}_{\mathsf{HB}}(r, x, w)$ generates the indices $I \subseteq [|r|]$ of r to reveal, along with a proof π
- **Verify:** $V_{\mathsf{HB}}(I, \{r_i\}_{i \in I}, \pi)$ outputs 1 if it accepts the proof and 0 otherwise

Such a proof system must satisfy completeness and soundness (similar to as defined earlier)

NIZK in Hidden-Bit Model (contd.)

Definition (NIZK in Hidden Bit Model)

A non-interactive proof system $(\mathsf{K}_{\mathsf{HB}}, \mathsf{P}_{\mathsf{HB}}, \mathsf{V}_{\mathsf{HB}})$ for a language L with witness relation R in the hidden-bit model is *(non-adaptive) zero-knowledge* if there exists a PPT simulator $\mathcal{S}_{\mathsf{HB}}$ s.t. for every $x \in L$, $w \in R(x)$, the output distributions of the following two experiments are computationally indistinguishable:

$\mathsf{REAL}(1^n, x, w)$	$\mathsf{IDEAL}(1^n, x)$
$r \leftarrow \mathsf{K}_{\mathsf{HB}}(1^n)$	$(I, \{r_i\}_{i \in I}, \pi) \leftarrow \mathcal{S}_{\mathsf{HB}}(1^n, x)$
$(I, \pi) \leftarrow \mathsf{P}_{\mathsf{HB}}(r, x, w)$	
Output $\left(I, \{r_i\}_{i \in I}, \pi\right)$	Output $\left(I, \{r_i\}_{i \in I}, \pi\right)$

From NIZK in HB Model to NIZK in CRS Model

Intuition: How to transform a "public" random string into a "hidden" random string

- Suppose the prover samples a trapdoor permutation (f, f^{-1}) with hardcore predicate h
- Given a common random string $\sigma = \sigma_1, \ldots, \sigma_n$, the prover can compute $r = r_1, \ldots, r_n$ where:

$$r_i = h(f^{-1}(\sigma_i))$$

- If f is a permutation and h is a hard-core predicate, then r is guaranteed to be random
- Now r can be treated as the hidden random string: V can only see the parts of it that the prover wishes to reveal

Construction

Let $\mathcal{F} = \{f, f^{-1}\}$ be a family of 2^n trapdoor permutations with hardcore predicate h. Let $(\mathsf{K}_{\mathsf{HB}}, \mathsf{P}_{\mathsf{HB}}, \mathsf{V}_{\mathsf{HB}})$ be a NIZK proof system for L in the hidden-bit model with soundness error 2^{-2n}

Construction of $(\mathsf{K}, \mathsf{P}, \mathsf{V})$**:**

$\mathsf{K}(1^n)$: Output a random string $\sigma = \sigma_1, \ldots, \sigma_n$ s.t. $\forall i, |\sigma_i| = n$

$\mathsf{P}(\sigma, x, w)$: Execute the following steps:

- Sample $(f, f^{-1}) \leftarrow \mathcal{F}(1^n)$
- Compute $\alpha_i = f^{-1}(\sigma_i)$ for $i \in [n]$
- Compute $r_i = h(\alpha_i)$ for $i \in [n]$
- Compute $(I, \varphi) \leftarrow \mathsf{P}_{\mathsf{HB}}(r, x, w)$
- Output $\pi = (f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$\mathsf{V}(\sigma, x, \pi)$: Parse $\pi = (f, I, \{\alpha_i\}_{i \in I}, \Phi)$ and:

- Check $f \in \mathcal{F}$ and $f(\alpha_i) = \sigma_i$ for every $i \in I$
- Compute $r_i = h(\alpha_i)$ for $i \in I$
- Output $\mathsf{V}_{\mathsf{HB}}(I, \{r_i\}_{i \in I}, x, \Phi)$

(K, P, V) is a Non-Interactive Proof

- **Completeness:** α is uniformly distributed since f^{-1} is a permutation and σ is random. Further, since h is a hard-core predicate, r is also uniformly distributed. Completeness follows from the completeness of $(\mathsf{K}_{\mathsf{HB}}, \mathsf{P}_{\mathsf{HB}}, \mathsf{V}_{\mathsf{HB}})$

- **Soundness:** For any $f = f_0$, r is uniformly random, so from (non-adaptive) soundness of $(\mathsf{K}_{\mathsf{HB}}, \mathsf{P}_{\mathsf{HB}}, \mathsf{V}_{\mathsf{HB}})$, we have:

$$\Pr_{\sigma}[P^* \text{ can cheat using } f_0] \leqslant 2^{-2n}$$

Since there are only 2^n possible choices of f (verifier checks that $f \in \mathcal{F}$), by union bound, it follows:

$$\Pr_{\sigma}[P^* \text{ can cheat }] \leqslant 2^{-n}$$

Proof of Zero Knowledge: Simulator

Let $\mathcal{S}_{\mathsf{HB}}$ be the simulator for $(\mathsf{K}_{\mathsf{HB}}, \mathsf{P}_{\mathsf{HB}}, \mathsf{V}_{\mathsf{HB}})$

Simulator $\mathcal{S}(1^n, x)$:

1. $(I, \{r_i\}_{i \in I}, \Phi) \leftarrow \mathcal{S}_{\mathsf{HB}}(1^n, x)$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in I$
4. $\sigma_i = f(\alpha_i)$ for every $i \in I$
5. $\sigma_i \xleftarrow{\$} \{0,1\}^n$ for every $i \notin I$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

Note: $h^{-1}(r_i)$ denotes sampling from the pre-image of r_i, which can be done efficiently by simply trying random α_i's until $h(\alpha_i) = r_i$

Proof of Zero Knowledge: Hybrids

Hybrid $H_0(1^n, x, w) := \mathsf{REAL}(1^n, x, w)$**:**

1. $\sigma \leftarrow \mathsf{K}(1^n)$ where $\sigma = \sigma_1, \ldots, \sigma_n$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow f^{-1}(\sigma_i)$ for every $i \in [n]$
4. $r_i = h(\alpha_i)$ for every $i \in [n]$
5. $(I, \Phi) \leftarrow \mathsf{P}_{\mathsf{HB}}(r, x, w)$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

Proof of Zero Knowledge: Hybrids (contd.)

Hybrid $H_1(1^n, x, w)$**:**

1. $\alpha_i \xleftarrow{\$} \{0,1\}^n$ for every $i \in [n]$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\sigma_i \leftarrow f(\alpha_i)$ for every $i \in [n]$
4. $r_i = h(\alpha_i)$ for every $i \in [n]$
5. $(I, \Phi) \leftarrow \mathsf{P}_{\mathsf{HB}}(r, x, w)$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$H_0 \approx H_1$: In H_1, we sample α_i at random and then compute σ_i (instead of sampling σ_i and then computing α_i as in H_0). This induces an identical distribution since f is a permutation

Proof of Zero Knowledge: Hybrids (contd.)

Hybrid $H_2(1^n, x, w)$**:**

1. $r_i \xleftarrow{\$} \{0,1\}$ for every $i \in [n]$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in [n]$
4. $\sigma_i = f(\alpha_i)$ for every $i \in [n]$
5. $(I, \Phi) \leftarrow \mathsf{P}_{\mathsf{HB}}(r, x, w)$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$H_1 \approx H_2$: In H_2, we again change the sampling order: first sample $r = r_1, \ldots, r_n$ at random and then sample α_i from the pre-image of r_i (as described earlier). This distribution is identical to H_1

Proof of Zero Knowledge: Hybrids (contd.)

Hybrid $H_3(1^n, x, w)$**:**

1. $r_i \xleftarrow{\$} \{0,1\}$ for every $i \in [n]$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in [n]$
4. $(I, \Phi) \leftarrow \mathsf{P}_{\mathsf{HB}}(r, x, w)$
5. $\sigma_i = f(\alpha_i)$ for every $i \in I$
6. $\sigma_i \xleftarrow{\$} \{0,1\}^n$ for every $i \notin I$
7. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$H_2 \approx_c H_3$: In H_3, we output random σ_i for $i \in I$. From security of hard-core predicate h, it follows that:

$$\{f(h^{-1}(r_i))\} \approx_c U_n$$

Indistinguishability of H_2 and H_3 follows using the above equation

Proof of Zero Knowledge: Hybrids (contd.)

Hybrid $H_4(1^n, x) := \mathsf{IDEAL}(1^n, x)$**:**

1. $(I, \{r_i\}_{i \in I}, \Phi) \leftarrow \mathcal{S}_{\mathsf{HB}}(1^n, x)$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in I$
4. $\sigma_i = f(\alpha_i)$ for every $i \in I$
5. $\sigma_i \xleftarrow{\$} \{0,1\}^n$ for every $i \notin I$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$H_3 \approx_c H_4$: In H_4, we swap P_{HB} with $\mathcal{S}_{\mathsf{HB}}$. Indistinguishability follows from the zero-knowledge property of $(\mathsf{K}_{\mathsf{HB}}, \mathsf{P}_{\mathsf{HB}}, \mathsf{V}_{\mathsf{HB}})$