One-way Functions

600.442: Modern Cryptography

Fall 2016

Today's Agenda

- Learning the crypto language
  - Modeling "real-world" adversaries
  - Defining security against such adversaries
- Definition of One-way functions
- Candidate One-way function

Modeling the adversary

- In practice, *everyone*, including the adversary has some bounded computational resources.
- Adversary can use these computational resources however intelligently he likes, but it is still bounded by these resources.
- Turing machines — capture all types of computations that are possible.
- So our adversary will be a computer program or an algorithm, modeled as a Turing machine.

Algorithms and Running Time

Definition (Algorithm)

An *algorithm* is a deterministic Turing machine whose input and output are strings over the binary alphabet $\Sigma = \{0, 1\}$.

Definition (Running Time)

An algorithm $\mathcal{A}$ is said to run in time $T(n)$ if for all $x \in \{0, 1\}^n$, $\mathcal{A}(x)$ halts within $T(|x|)$ steps. $\mathcal{A}$ runs in polynomial time if there exists a constant c such that $\mathcal{A}$ runs in time $T(n) = n^c$.

An algorithm is *efficient* if it runs in polynomial time.

Randomized Algorithms

Definition (Randomized Algorithm)

A *randomized algorithm*, also called a probabilistic polynomial time Turing machine (PPT) is a Turing machine equipped with an extra *randomness* tape. Each bit of the randomness tape is uniformly and independently chosen.

- Output of a randomized algorithm is a distribution.
- This notion captures what we can do efficiently *ourselves*. (uniform TMs)

The Adversary

- The adversary could be more tricky...
- For example, the adversary might posses a *different* algorithm for each input size, each of which might be efficient.
- This still counts efficient since he is using polynomial time resources!
- We call this a *non-uniform* adversary since the algorithm is not uniform across all input sizes.

Non-Uniform PPT

Definition (Non-Uniform PPT)

A *non-uniform probabilistic polynomial time Turing machine* is a Turing machine A is a sequence of probabilistic machines $A = \{A_1, A_2, \ldots\}$ for which there exists a polynomial $p(\cdot)$ such that for every $A_i \in A$, the description size $|A_i|$ and the running time of A_i are at most $p(i)$. We write $A(x)$ to denote the distribution obtained by running $A_{|x|}(x)$.

- Our adversary will usually be a non-uniform PPT Turing machine. (most general)

One Way Functions: Attempt 1

Attempt 1: A function $f : \{0,1\}^* \to \{0,1\}^*$ is a one-way function (OWF) if it satisfies the following two conditions:

- **Easy to compute:** there is a PPT algorithm $\mathcal{C}$ s.t. $\forall x \in \{0,1\}^*$,

$$\Pr\left[\mathcal{C}(x) = f(x)\right] = 1.$$

- **Hard to invert:** for every non-uniform PPT adversary $\mathcal{A}$, for any input length $n \in \mathbb{N}$

Probability of Inversion is small

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$$\Pr\left[\mathcal{A} \text{ inverts } f(x) \text{ for random } x\right] \leqslant \textit{small}.$$

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- **Hard to invert:** for every non-uniform PPT adversary $\mathcal{A}$, there exists a fast decaying function $\nu(\cdot)$ s.t. for any input length $n \in \mathbb{N}$

$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n;\ \mathcal{A} \text{ inverts } f(x)\right] \leqslant \nu(n).$$

Negligible Function

Definition (Negligible Function)

A function $\nu(n)$ is negligible if for every c, there exists some n_0 such that for all $n > n_0$, $\nu(n) \leqslant \frac{1}{n^c}$.

1. Negligible function decays faster than all "inverse-polynomial" functions
2. That is, $n^{-\omega(1)}$

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$$\Pr\left[\mathcal{C}(x) = f(x)\right] = 1.$$

- **Hard to invert:** for every non-uniform PPT adversary $\mathcal{A}$, there exists a *negligible* function $\mu(\cdot)$ s.t. for any input length $\forall n \in \mathbb{N}$:

$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n;\ \mathcal{A} \text{ inverts } f(x)\right] \leqslant \nu(|x|).$$

Technical Problem: What is $\mathcal{A}$'s input?

$\mathcal{A}$'s Input

- Let's write $y = f(x)$.
- **Condition 1:** $\mathcal{A}$ on input y must run in time $\text{poly}(|y|)$.
- **Condition 2:** $\mathcal{A}$ cannot output x' s.t. $f(x') = y$.
- What if $|y|$ is much smaller than $n = |x|$?
  $\implies$ $\mathcal{A}$ cannot write the inverse even if it can find it!

- Example: $f(x) =$ first $\log |x|$ bits of x.
- It is trivial to invert: $f^{-1}(y) = y \| \underbrace{00\ldots0}_{n-\lg n}$ where $n = 2^{|y|}$.
- But it satisfies our Attempt 1 defintion!
  - f is easy to compute.
  - $\mathcal{A}$ cannot invert in time $\text{poly}(|y|)$.
    It needs $2^{|y|}$ steps just to write the answer!

Fixing the definition

- Give $\mathcal{A}$ a long enough input.
- If y is too short, pad it with $1s$ in the beginning.
- We adopt the convention to **always** pad it and write: $\mathcal{A}(1^n, y)$.
- Now $\mathcal{A}$ has enough time to write the answer.

One Way Functions: Definition

Definition (One Way Function)

A function $f : \{0,1\}^* \to \{0,1\}^*$ is a *one-way function* (OWF) if it satisfies the following two conditions:

- **Easy to compute:** there is a PPT algorithm $\mathcal{C}$ s.t. $\forall x \in \{0,1\}^*$,

$$\Pr\left[\mathcal{C}(x) = f(x)\right] = 1.$$

- **Hard to invert:** there exists a *negligible* function $\mu : \mathbb{N} \to \mathbb{R}$ s.t. for every non-uniform PPT adversary $\mathcal{A}$ and $\forall n \in \mathbb{N}$:

$$\Pr\left[x \leftarrow \{0,1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') = f(x)\right] \leqslant \mu(n).$$

This definition is also called **strong** one-way functions.

Injective OWFs and One Way Permutations (OWP)

- **Injective or 1-1 OWFs**: each image has a *unique* pre-image:

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

- **One Way Permutations (OWP)**: 1-1 OWF with the additional conditional that "each image has a pre-image"

  (Equivalently: domain and range are of same size.)

Existence of OWFs

- Do OWFs exist? NOT Unconditionally — proving that f is one-way requires proving (at least) $\mathbf{P} \neq \mathbf{NP}$.
- However, we can construct them ASSUMING that certain problems are hard.
- Such constructions are sometimes called "candidates" because they are based on an assumption or a conjecture.

Factoring Problem

- Consider the **multiplication** function $f_\times : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$:

$$f_\times(x, y) = \begin{cases} \perp & \text{if } x = 1 \vee y = 1 \\ x \cdot y & \text{otherwise} \end{cases}$$

- The first condition helps exclude the trivial factor 1.
- Is $f_\times$ a OWF?
- **Clearly not!** With prob. 1/2, a random number (of any fixed size) is *even*. I.e., xy is even w/ prob. $\frac{3}{4}$ for random (x, y).
- Inversion: given number z, output $(2, z/2)$ if z is even and $(0, 0)$ otherwise!(succeeds 75% time)

Factoring Problem (continued)

- Eliminate such trivial small factors.
- Let Π_n be the set of all **prime** numbers $< 2^n$.
- Choose numbers p and q randomly from Π_n and multiply.
- This is unlikely to have small trivial factors.

Assumption (Factoring Assumption)

For every (non-uniform PPT) adversary $\mathcal{A}$, there exists a negligible function ν such that

$$\Pr\left[p \xleftarrow{\$} \Pi_n; q \xleftarrow{\$} \Pi_n; N = pq : \mathcal{A}(N) \in \{p, q\}\right] \leqslant \nu(n).$$

Factoring Problem (continued)

- Factoring assumption is a well established conjecture.
- Studied for a long time, with no "good" attack.
- Best known algorithms for breaking Factoring Assumption:

$$
\begin{array}{ll}
2^{O\left(\sqrt{n \log n}\right)} & \text{(provable)} \\
2^{O\left(\sqrt[3]{n \log^2 n}\right)} & \text{(heuristic)}
\end{array}
$$

- Can we construct OWFs from the Factoring Assumption?

Back to Multiplication Function

- Let's reconsider the function $f_\times : \mathbb{N}^2 \to \mathbb{N}$.
- Clearly, if a random x and a random y happen to be prime, no $\mathcal{A}$ could invert. Call it the GOOD case.
- If GOOD case occurs with probability $> \varepsilon$,

  $\Rightarrow$ every $\mathcal{A}$ must fail to invert $f_\times$ with probability at least ε.
- Now suppose that ε is a noticeable function

  $\Rightarrow$ every $\mathcal{A}$ must fail to invert $f_\times$ with noticeable probability.
- This is already useful!
- Usually called a **weak** OWF.

Weak One Way Functions

Definition (Weak One Way Function)

A function $f : \{0,1\}^* \to \{0,1\}^*$ is a *weak one-way function* if it satisfies the following two conditions:

- **Easy to compute:** there is a PPT algorithm $\mathcal{C}$ s.t. $\forall x \in \{0,1\}^*$,

$$\Pr\left[\mathcal{C}(x) = f(x)\right] = 1.$$

- **Somewhat hard to invert:** there is a noticeable function $\varepsilon : \mathbb{N} \to \mathbb{R}$ s.t. for every non-uniform PPT $\mathcal{A}$ and $\forall n \in \mathbb{N}$:

$$\Pr\left[x \leftarrow \{0,1\}^n, x' \leftarrow \mathcal{A}(1^n, f(x)) : f(x') \neq f(x)\right] \geqslant \varepsilon(n).$$

Noticeable means $\exists c$ s.t. for infinitely many $n \in \mathbb{N}$, $\varepsilon(n) \geqslant \frac{1}{n^c}$.

Back to Multiplication

- Can we prove that $f_\times$ is a weak OWF?
- Remember the GOOD case? Both x and y are prime.
- If we can show that GOOD case occurs with noticeable probability, we can prove that $f_\times$ is a weak OWF.

Theorem

Assuming the factoring assumption, function $f_\times$ is a weak OWF.

- Proof Idea: The fraction of prime numbers between 1 and 2^n is noticeable!
- Chebyshev's theorem: An n bit number is a prime with probability $\frac{1}{2n}$

What about normal OWFs?

- Can we construct normal (a.k.a, strong) OWFs from the Factoring Assumption?
- Even better:
  Can we construction strong OWFs from ANY weak OWF?
- Yes! Yao's theorem.

Weak to Strong OWFs

Theorem (Yao)

Strong OWFs exist if and only weak OWFs exist.

- This is called hardness amplification: convert a somewhat hard problem into a really hard problem.
- Hint: use **many** samples of the weak OWF as the output of the strong OWF.
- Proof by reduction: if $\mathcal{A}$ can break your strong OWF, you can come up with an algorithm $\mathcal{B}$ for breaking weak OWFs.